Tutorial sheet 7

Discussion topic: Onsager relations & fluctuation-dissipation theorem in linear response theory

14. Detailed balance relation

Consider a two-level system coupled to a macroscopic system Σ at thermodynamic equilibrium. The total Hamiltonian is given by

$$\hat{H} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hat{H}_{\rm int} + \hat{H}_\Sigma \equiv \hat{H}_0 + \hat{H}_{\rm int}, \tag{1}$$

with σ_z the usual Pauli matrix. Let $|g\rangle$, $|e\rangle$ denote the two levels (ground state, excited state) of the small system and $\{\,|\phi_n\rangle\}$ be a basis of energy eigenstates of $\hat{H}_\Sigma$. The interaction between the systems is chosen so as to induce transitions in the small one:

$$\hat{H}_{\rm int} = -\hat{\sigma}_x \otimes \hat{X}, \tag{2}$$

with $\hat{X}$ an observable of the macroscopic system.

We want to study the absorption and emission rates $\Gamma_{g\to e}$, $\Gamma_{e\to g}$ of the small system.

i. Time evolution

Let $|\psi(0)\rangle = |\alpha\rangle \otimes |\phi_n\rangle$ with $\alpha = g$ or e denote the initial state of the composite system. Check that to lowest non-trivial order in perturbation theory this state becomes at time t

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}_0 t/\hbar}\left[1 + \frac{\mathrm{i}}{\hbar}\int_0^t \hat{\sigma}_x(t')\hat{X}(t')\,\mathrm{d}t'\right]|\psi(0)\rangle.$$

ii. Absorption rate

a) Deduce from the previous result that the absorption probability at time t when the macroscopic system is initially in the state $|\phi_n\rangle$ is given by

$$p_{\rm abs}^{(\phi_n)}(t) = \sum_m \left|\big(\langle e|\otimes\langle\phi_m|\big)\,|\psi(t)\rangle\right|^2 \simeq \frac{1}{\hbar^2}\int_0^t\!\!\int_0^t \langle\phi_n|\,\hat{X}(t')\hat{X}(t'')\,|\phi_n\rangle\,\mathrm{e}^{-\mathrm{i}\omega_0(t'-t'')}\,\mathrm{d}t'\,\mathrm{d}t''.$$

Hint: It might help to realize that $\hat{\sigma}_x(t) = \frac{1}{2}(\hat{\sigma}_+\,\mathrm{e}^{-\mathrm{i}\omega_0 t} + \hat{\sigma}_-\,\mathrm{e}^{\mathrm{i}\omega_0 t})$, from where its matrix elements become trivial, with $\sigma_\pm$ the usual linear combinations of σ_x and σ_y.

b) Check that this leads for the total absorption probability to

$$p_{\rm abs}(t) \simeq \frac{1}{\hbar^2}\int_0^t\!\!\int_0^t \big\langle\hat{X}(t')\hat{X}(t'')\big\rangle_{\rm eq.}\,\mathrm{e}^{-\mathrm{i}\omega_0(t'-t'')}\,\mathrm{d}t'\,\mathrm{d}t'',$$

where $\langle\,\cdot\,\rangle_{\rm eq.}$ denotes the usual expectation value at equilibrium using the density operator of the macroscopic system.

c) One recognizes in the integrand the non-symmetrized correlation function $C_{XX}(t'-t'')$ for the observable of the system Σ. Assuming that this function only takes significant values when its argument is close to 0, show how you can replace the double integral according to

$$\int_0^t\!\!\int_0^t \mathrm{d}t'\,\mathrm{d}t'' \simeq t\int_{-\infty}^{\infty}\mathrm{d}(t'-t'').$$

Deduce from there the absorption rate

$$\Gamma_{g\to e} = \frac{1}{\hbar^2}\tilde{C}_{XX}(-\omega_0), \tag{3}$$

with $\tilde{C}_{XX}$ the Fourier transform of C_{XX}.

iii. Emission rate

Repeating the computation of question **ii.** (you need not do it explicitly! the only change is that of a matrix element), deduce that the emission rate of the small system is

$$\Gamma_{e\to g} = \frac{1}{\hbar^2}\tilde{C}_{XX}(\omega_0). \tag{4}$$

Writing now the

iv. Detailed balance relation for $\tilde{C}_{XX}$ gives you a simple relationship between the absorption and emission rates.

We now let the populations π_g, π_e of the two levels according to the coupled *rate equations* (do you agree that they make sense if one remains in a linear regime?)

$$\frac{\mathrm{d}\pi_g(t)}{\mathrm{d}t} = -\Gamma_{g\to e}\pi_g(t) + \Gamma_{e\to g}\pi_e(t)$$
$$\frac{\mathrm{d}\pi_e(t)}{\mathrm{d}t} = \Gamma_{g\to e}\pi_g(t) - \Gamma_{e\to g}\pi_e(t).$$

Find the ratio π_e/π_g of these populations in the stationary regime: first in terms of the rates; then, using the detailed balance relation, in terms of the energy difference between both levels. What do you recognize?

If you want more information on the possible interest of the system modelled above, you can have a look at R. J. Schoelkopf *et al.*, *Qubits as Spectrometers of Quantum Noise*, `http://arxiv.org/abs/cond-mat/0210247`.

15. Classical linear response

Read the section on classical linear response in the lecture notes — it should be uploaded to the web page of the course by the end of the week.