Tutorial sheet 14

28. Bulk viscosity

In a hydrodynamic model, the (conserved) stress-energy tensor for a fluid obeying the Navier–Stokes equation can be written as

$$T_{ij}(t,\vec{r}) = \mathcal{P}(t,\vec{r})\delta_{ij} + \pi_{ij}^{\rm shear}(t,\vec{r}) + \Pi(t,\vec{r})\delta_{ij},$$

where $\mathcal{P}$ is the equilibrium pressure, $\pi_{ij}^{\rm shear}$ the traceless shear stress tensor—which depends on the first derivatives of the flow velocity $\vec{\mathsf{v}}(t,\vec{r})$—, and Π the bulk pressure. The latter is of the form $\Pi = \zeta\vec{\nabla}\cdot\vec{\mathsf{v}}$, with ζ the bulk viscosity. Thus, ζ measures the size of the deviation [in compressible ($\vec{\nabla}\cdot\vec{\mathsf{v}} \neq 0$) flows] of the trace $\sum_i T_{ii}$ of the stress-energy tensor from the equilibrium value $3\mathcal{P}$.

Microscopically, the stress-energy tensor for a gas of non-interacting particles is related to the phase space density $\bar{f}(t,\vec{r},\vec{p})$ through

$$T_{ij}(t,\vec{r}) \equiv \int p_i p_j \bar{f}(t,\vec{r},\vec{p})\,\frac{{\rm d}^3\vec{p}}{(2\pi\hbar)^3 E_{\vec{p}}},$$

where $E_{\vec{p}}$ denotes the energy of a particle with momentum $\vec{p}$, both quantities being measured in the local rest frame of the fluid — i.e. the momentum-space average of $\vec{p}$, weighted with $\bar{f}$, vanishes.

Similarly, the energy density e—which is related to the equilibrium pressure with the help of the equation of state of the fluid—is given by

$$e(t,\vec{r}) \equiv \int E_{\vec{p}}\bar{f}(t,\vec{r},\vec{p})\,\frac{{\rm d}^3\vec{p}}{(2\pi\hbar)^3} = \int E_{\vec{p}}^2\bar{f}(t,\vec{r},\vec{p})\,\frac{{\rm d}^3\vec{p}}{(2\pi\hbar)^3 E_{\vec{p}}},$$

where the second identity was included to emphasize the analogy with the definition of T_{ij}.

i. Gas of massless particles

For a noninteracting gas of massless particles, the equation of state reads $e = 3\mathcal{P}$.[1] Using the known relation for $E_{\vec{p}}$ as function of momentum, check that, *irrespective of the expression for* $\bar{f}$, the trace of the stress-energy tensor simply equals e. What does this mean for the bulk viscosity ζ?

ii. Nonrelativistic ideal gas

The internal energy U and pressure $\mathcal{P}$ of a nonrelativistic ideal gas of particles with mass m occupying a volume $\mathcal{V}$ obey the relation $U = \frac{3}{2}\mathcal{P}\mathcal{V}$.[1] Using the nonrelativistic expression for $E_{\vec{p}}$ and appropriate approximations, compute the trace of the stress-energy tensor and express it as a (simple) function of the energy density e and the particle number density n defined as the integral over momenta of the phase space density. What does this again give for the bulk viscosity ζ?

Hint: You should recover the expression of the stress-energy tensor given in the lecture, where it is denoted by P_{ij}... Remember also what is included in the internal energy U, and what is not.

Remark: Since no knowledge of the phase space density $\bar{f}$ was actually needed in the above derivations, the systems need not be equilibrated for the result(s) to hold.

29. Validity of the hydrodynamic regime

Consider a Boltzmann gas with typical density n, contained in a volume L^3. Let $\ell_{\rm mfp}$ denote the mean free path of the particles in the gas. A (crude) criterion for the validity of the hydrodynamic

[1] Check your favorite course on equilibrium statistical physics and thermostatics!

description of the gas motion is that the *Knudsen number* $\mathrm{Kn} \equiv \ell_{\mathrm{mfp}}/L$ should be (much) smaller than unity. Do you see why? How can you interpret Kn^{-1}?

Which other two (strong) inequalities should the three parameters of the "Boltzmann gas" obey? Are all three conditions compatible?

30. Relativistic kinetic theory

Educate yourself on the relativistic version of the Boltzmann kinetic equation, by reading Chapter I (& II?) of the book by de Groot, van Leeuwen & van Weert (link available on the lecture web page).