Tutorial sheet 13

Discussion topic: Boltzmann equation

26. Electrical conductivity in a magnetic field[1]

We consider the problem of electric conduction in a metal subject to a constant and uniform electromagnetic field ($\vec{\mathscr{E}}$,$\vec{\mathscr{B}}$). The conduction electrons (mass $m_{\rm e}$, charge $-e$) form a non-relativistic, highly degenerate ideal Fermi gas obeying the kinetic Lorentz equation (cf. Tutorial sheet 12) in the presence of an external force $\vec{F}$, which is here simply the Lorentz force. We assume that the various densities are uniform and stationary: $\bar{f}(t,\vec{r},\vec{p}) = \bar{f}(\vec{p})$, where $\vec{p}$ denotes the linear momentum, and $n(t,\vec{r}) = n$, which leads to simplifications in the left-hand side of the Lorentz equation. In addition, we assume that the local equilibrium distribution is a function of energy only: $\bar{f}_{(0)}(\vec{p}) = \bar{f}_{(0)}(\varepsilon)$ with $\varepsilon \equiv \vec{p}^2/2m_{\rm e}$.

i. We first take $\vec{\mathscr{B}} = \vec{0}$. Calculate $\delta\bar{f} \equiv \bar{f} - \bar{f}_{(0)}$ and show that the electric current density $\vec{J}_{\rm el.}$ is given in the relaxation time approximation by

$$\vec{J}_{\rm el.} = -e^2 \int \tau_{\rm r}(|\vec{p}|)(\vec{v}\cdot\vec{\mathscr{E}})\vec{v}\,\frac{{\rm d}\bar{f}_{(0)}}{{\rm d}\varepsilon}\,\frac{{\rm d}^3\vec{p}}{(2\pi\hbar)^3}.$$

Show that, if the local equilibrium distribution is the Fermi distribution (with 2 spin degrees of freedom) at $T = 0$, $\bar{f}_{(0)}(\varepsilon) = 2\,\Theta(\varepsilon_{\rm F} - \varepsilon)$ with $\varepsilon_{\rm F}$ the Fermi energy, then $\vec{J}_{\rm el.}$ obeys Ohm's law with the electrical conductivity

$$\sigma_{\rm el.} = \frac{n\,e^2}{m_{\rm e}}\tau_{\rm F}$$

where $\tau_{\rm F} \equiv \tau_{\rm r}(p_{\rm F})$, with $p_{\rm F}$ the Fermi momentum.

ii. Let now $\vec{\mathscr{B}} \neq \vec{0}$. How is the electrical conductivity modified if $\vec{\mathscr{B}}$ is parallel to $\vec{\mathscr{E}}$?

iii. Consider the case where the electric field is in the xy-plane and the magnetic field along the z-axis, $\vec{\mathscr{B}} = \mathscr{B}\,\vec{\rm e}_z$ with $\mathscr{B} > 0$.

a) Show that the Lorentz equation in the relaxation time approximation becomes

$$-e\vec{v}\cdot\vec{\mathscr{E}}\,\frac{{\rm d}\bar{f}_{(0)}}{{\rm d}\varepsilon} - e\big(\vec{v}\times\vec{\mathscr{B}}\big)\cdot\vec{\nabla}_{\vec{p}}\delta\bar{f} = -\frac{\delta\bar{f}}{\tau_r(|\vec{p}|)}.$$

b) We look for a solution of the form

$$\delta\bar{f} = -\vec{v}\cdot\vec{C}\,\frac{{\rm d}\bar{f}_{(0)}}{{\rm d}\varepsilon}$$

with $\vec{C}$ a vector, function of $\vec{\mathscr{E}}$ and $\vec{\mathscr{B}}$ but independent of $\vec{v}$, to be determined. What should $\vec{C}$ be when $\vec{\mathscr{B}} = \vec{0}$? when $\vec{\mathscr{E}} = \vec{0}$? For the latter case, estimate first the average magnetic force on the electrons.

c) Show that $\vec{C}$ satisfies the equation

$$-e\vec{\mathscr{E}} + \vec{\omega}\times\vec{C} = \frac{\vec{C}}{\tau_r(|\vec{p}|)},$$

with $\vec{\omega} = \omega\,\vec{\rm e}_z$, where $\omega \equiv e\mathscr{B}/m_{\rm e}$ is the Larmor frequency. Justify that $\vec{C}$ is necessarily of the form $\vec{C} = \alpha\vec{\mathscr{E}} + \delta\vec{\mathscr{B}} + \gamma\vec{\mathscr{B}}\times\vec{\mathscr{E}}$, where α,δ,γ are real numbers. Find the expression for $\vec{C}$ and show that

$$\delta\bar{f} = \frac{e\tau_{\rm r}}{1+\omega^2\tau_{\rm r}^2}\big(\vec{\mathscr{E}} + \tau_{\rm r}\vec{\omega}\times\vec{\mathscr{E}}\big)\cdot\vec{v}\,\frac{{\rm d}\bar{f}_{(0)}}{{\rm d}\varepsilon}.$$

[1] This exercise was shamelessly stolen from the book *Equilibrium and non-equilibrium statistical thermodynamics* by M. Le Bellac *et al.*

d) Calculate the electric current and the components $(\sigma_{\text{el.}})_{ij}$, $i,j=x,y$ of the electrical conductivity tensor. Verify that $(\sigma_{\text{el.}})_{xy} = -(\sigma_{\text{el.}})_{yx}$ and comment on this relation.

27. Electrical conductivity in a magnetic field: Hall effect

This exercise is a sequel to the previous one, yet tackles the problem differently. We again consider the problem of electric conduction in a metal. We now assume that the conductor subject to the electric and magnetic fields $\vec{\mathscr{E}}$, $\vec{\mathscr{B}}$ is a rectangular parallelepiped, with its sides along the coordinate axes. Let L, l, and d be the respective lengths of the sides parallel to the x-, y and z-directions.

i. Drude–Lorentz model
To model the effect of collisions in a simple way, one introduces an "average equation of motion" for the conduction electrons—i.e., an evolution equation for their average velocity $\langle\vec{v}\rangle$

$$\frac{\mathrm{d}\langle\vec{v}\rangle}{\mathrm{d}t} = -\frac{\langle\vec{v}\rangle}{\tau_{\mathrm{r}}} - \frac{e}{m_{\mathrm{e}}}\big(\vec{\mathscr{E}} + \langle\vec{v}\rangle \times \vec{\mathscr{B}}\big).$$

Give a physical interpretation for this equation. Check that in the stationary regime one has

$$\langle v_x\rangle = -\frac{e\tau_{\mathrm{r}}}{m_{\mathrm{e}}}\,\mathscr{E}_x - \omega\tau_{\mathrm{r}}\langle v_y\rangle\,, \qquad \langle v_y\rangle = -\frac{e\tau_{\mathrm{r}}}{m_{\mathrm{e}}}\,\mathscr{E}_y + \omega\tau_{\mathrm{r}}\langle v_x\rangle\,,$$

with ω the Larmor frequency defined in exercise 26. Show that if one takes $\tau_{\mathrm{r}} = \tau_{\mathrm{F}}$, one recovers the same expression for the conductivity tensor as in exercise 26.

ii. Calculate in terms of $\mathscr{E}_x$ the value $\mathscr{E}_{\mathrm{H}}$ of $\mathscr{E}_y$ which cancels $(J_{\text{el.}})_y$. Verify that the transport of electrons in that situation is the same as in the case $\vec{\mathscr{B}} = \vec{0}$, in other words $(J_{\text{el.}})_x = \sigma_{\text{el.}}\mathscr{E}_x$. The field intensity $\mathscr{E}_{\mathrm{H}}$ is called *Hall field*, and the *Hall resistance* is defined by

$$R_{\mathrm{H}} \equiv \frac{V_{\mathrm{H}}}{I}$$

where V_{H} is the *Hall voltage*, $V_{\mathrm{H}}/l = \mathscr{E}_{\mathrm{H}}$, and I the total electric current along the x direction. Show that R_{H} is given by

$$R_{\mathrm{H}} = \frac{\mathscr{B}}{n de},$$

with n the density of conduction electrons. By noting that R_{H} is independent of the relaxation time, find its expression using an elementary argument.