Tutorial sheet 12

Discussion topic: Boltzmann equation: equilibrium distributions; relaxation-time approximation

25. Lorentz gas

In various physical situations—for instance the motion of neutrons in a nuclear reactor or of electrons in a non-degenerate semi-conductor—one can describe the diffusion of (light) particles inside a medium made of more massive constituents as resulting from collisions on *fixed* scattering centres. The diffusing particles are then referred to as a *Lorentz gas*. The evolution of the corresponding (coarse-grained) single-particle phase-space density is governed by a simplified version of the Boltzmann kinetic equation, which will be derived hereafter.

Throughout this problem, it is assumed that no external vector potential is present, so that the linear and canonical momenta $\vec{p}$ of particles are identical. Furthermore, one neglects the collisions between the particles of the Lorentz gas.

i. Conservation laws

One assumes that the collisions between the particles and the scattering centres are instantaneous and local, as well as elastic and invariant under space parity and time reversal.
Write down the relevant conservation laws. Explain why the collision of a particle on a scattering centre amounts to a change $\vec{p} \to \vec{p}\,'$ of its momentum, with $|\vec{p}| = |\vec{p}\,'|$. The corresponding differential cross-section will be denoted as $\sigma(\vec{p} \to \vec{p}\,')$. What can you say about reference frames?

ii. Kinetic equation

Let $\bar{f}(t, \vec{r}, \vec{p})$ be the (dimensionless) single-particle phase-space density of the particles of the Lorentz gas. As in the case of the Boltzmann equation, the dynamics of $\bar{f}$ obeys an equation of the type

$$\frac{\partial \bar{f}}{\partial t} + \vec{v} \cdot \vec{\nabla}_{\vec{r}} \bar{f} + \vec{F} \cdot \vec{\nabla}_{\vec{p}} \bar{f} = \left(\frac{\partial \bar{f}}{\partial t}\right)_{\rm coll.}, \qquad \left(\frac{\partial \bar{f}}{\partial t}\right)_{\rm coll.} \equiv \left(\frac{\partial \bar{f}}{\partial t}\right)_{\rm gain} - \left(\frac{\partial \bar{f}}{\partial t}\right)_{\rm loss}, \tag{1}$$

where $\vec{v}$ is the particle velocity, while the collision integral is conveniently expressed as the difference between a gain and a loss term, which we now want to compute.

a) Loss term. Consider a scattering centre at position $\vec{r}$. What is the flux density of particles with a momentum between $\vec{p}$ and $\vec{p} + {\rm d}^3\vec{p}$ falling on this scattering centre? Show that the number of collisions of the type $\vec{p} \to \vec{p}\,'$, where $\vec{p}$, $\vec{p}\,'$ are known up to infinitesimal uncertainties ${\rm d}^3\vec{p}$, ${\rm d}^3\vec{p}\,'$, in a time interval ${\rm d}t$ is given by

$$\bar{f}(t, \vec{r}, \vec{p})\, \frac{{\rm d}^3\vec{p}}{(2\pi\hbar)^3} \big|\vec{v}\big|\, \sigma(\vec{p} \to \vec{p}\,')\, {\rm d}^2\Omega'\, {\rm d}t,$$

where Ω' is the solid angle associated with the direction of $\vec{p}\,'$.
Deduce that the loss term is given by

$$\left(\frac{\partial \bar{f}}{\partial t}\right)_{\rm loss} = n_{\rm d}(\vec{r})\, \big|\vec{v}\big|\, \bar{f}(t, \vec{r}, \vec{p}) \int \sigma(\vec{p} \to \vec{p}\,')\, {\rm d}^2\Omega',$$

with $n_{\rm d}$ the density of scattering centres. What is the integral equal to?

b) Gain term. The gain term corresponds to collisions of the type $\vec{p}\,' \to \vec{p}$, where the initial and final momenta are known up to infinitesimal uncertainties. Show that between t and $t + {\rm d}t$, there are

$$\bar{f}(t, \vec{r}, \vec{p}\,')\, \frac{{\rm d}^3\vec{p}\,'}{(2\pi\hbar)^3} \big|\vec{v}\,'\big|\, \sigma(\vec{p}\,' \to \vec{p})\, {\rm d}^2\Omega\, {\rm d}t,$$

such collisions on a single scattering centre at position $\vec{r}$, where Ω is the solid angle associated with $\vec{p}$.

Let Ω' be the solid angle associated with $\vec{p}\,'$. Justify the identity

$$\left|\vec{v}\,'\right| \mathrm{d}^3\vec{p}\,' \,\mathrm{d}^2\Omega = \left|\vec{v}\right| \mathrm{d}^3\vec{p}\,\mathrm{d}^2\Omega'.$$

Show that the gain term can be written as

$$\left(\frac{\partial\bar{f}}{\partial t}\right)_{\text{gain}} = n_{\mathrm{d}}(\vec{r}\,)\left|\vec{v}\right| \int \bar{f}(t,\vec{r},\vec{p}\,')\,\sigma(\vec{p}\to\vec{p}\,')\,\mathrm{d}^2\Omega',$$

and thus the collision integral as

$$\left(\frac{\partial\bar{f}}{\partial t}\right)_{\text{coll.}} = n_{\mathrm{d}}(\vec{r}\,)\left|\vec{v}\right| \int \left[\bar{f}(t,\vec{r},\vec{p}\,') - \bar{f}(t,\vec{r},\vec{p}\,)\right]\sigma(\vec{p}\to\vec{p}\,')\,\mathrm{d}^2\Omega'. \tag{2}$$

iii. Diffusion coefficient of a Lorentz gas

Consider now a Lorentz gas at uniform temperature T, in which a time-independent density gradient is applied. The corresponding local-equilibrium solution that cancels the collision integral (2) is

$$\bar{f}_{(0)}(\vec{r},\vec{p}\,) = n(\vec{r}\,)\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} \mathrm{e}^{-\vec{p}^{\,2}/2mk_BT},$$

with $n(\vec{r}\,)$ the particle number density and m the mass of the particles.

In the relaxation-time approximation, the collision integral is rewritten as $-\left[\bar{f}(t,\vec{r},\vec{p}\,) - \bar{f}_{(0)}(\vec{r},\vec{p}\,)\right]/\tau_{\mathrm{r}}(\vec{p}\,)$, where $\bar{f}$ remains close to $\bar{f}_{(0)}$:

$$\bar{f}(t,\vec{r},\vec{p}\,) = \bar{f}_{(0)}(\vec{r},\vec{p}\,) + \bar{f}_{(1)}(t,\vec{r},\vec{p}\,), \qquad \left|\bar{f}_{(1)}\right| \ll \bar{f}_{(0)}.$$

We shall investigate the stationary phase-space density in this approximation, assuming that the relaxation time τ_{r} is momentum-independent.

a) Show that to first order and in the absence of external force $\vec{F}$, $\bar{f}_{(1)}$ obeys a differential equation whose stationary solution is

$$\bar{f}_{(1)}(\vec{r},\vec{p}\,) = -\tau_{\mathrm{r}}\,\vec{v}\cdot\vec{\nabla}_{\vec{r}}\bar{f}_{(0)}(\vec{r},\vec{p}\,). \tag{3}$$

b) The stationary particle current density for the constituents of the Lorentz gas is

$$\vec{\mathcal{J}}_N(\vec{r}\,) = \int \bar{f}(\vec{r},\vec{p}\,)\,\vec{v}\,\frac{\mathrm{d}^3\vec{p}}{(2\pi\hbar)^3}.$$

Expressing the stationary solution (3) as a function of the (spatial!) gradient of $n(\vec{r}\,)$, show that one recovers Fick's law of diffusion, with the diffusion coefficient $D = k_BT\tau_{\mathrm{r}}/m$.