Tutorial sheet 9

14. Fokker–Planck equation as approximation to the master equation

As seen in the lecture, the evolution of the probability density $p_{Y,1}(\tau, y)$ of an homogeneous Markov process $Y(t)$ is governed by the master equation

$$\frac{\partial p_{Y,1}(\tau, y)}{\partial \tau} = \int \big[\Gamma(y \,|\, y')\, p_{Y,1}(\tau, y') - \Gamma(y' \,|\, y)\, p_{Y,1}(\tau, y)\big]\, \mathrm{d}y',$$

where $\Gamma(y \,|\, y')$ denotes the transition rate from y' to y. In many situations, only small jumps $w \equiv y - y'$ occur. Rewriting $W(y', w) \equiv \Gamma(y' + w \,|\, y')$, this means that $W(y', w)$ is a sharply peaked function of w, while it varies slowly with y'. Assuming that $p_{Y,1}(\tau, y)$ also varies slowly with y, show that a Taylor expansion of the master equation up to second order yields the Fokker–Planck equation

$$\frac{\partial p_{Y,1}(\tau, y)}{\partial \tau} = -\frac{\partial}{\partial \tau}\big[M_1(y) p_{Y,1}(\tau, y)\big] + \frac{1}{2}\frac{\partial^2}{\partial \tau^2}\big[M_2(y) p_{Y,1}(\tau, y)\big] \quad \text{with} \quad M_n(y) \equiv \int w^n\, W(y, w)\, \mathrm{d}w.$$

15. Another view of the Fokker–Planck equation in one dimension

Consider an arbitrary one-dimensional Markovian process $X(t)$, taking its values in a real interval $[a, b]$, and such that the corresponding first two coefficients $M_1(t, x)$, $M_2(t, x)$ in the Kramers–Moyal expansion are actually independent of time.

i. Stationary solutions

Recall the form of the Fokker–Planck equation. Assuming that there is no flow of probability across the boundaries $x = a$ and $x = b$ (“reflecting boundary conditions”), write down the differential equation obeyed by the stationary solution $p^{\text{st.}}_{X,1}(x)$ to the Fokker–Planck equation. Show that

$$p^{\text{st.}}_{X,1}(x) = \frac{C}{M_2(x)} \exp\left[2\int_a^x \frac{M_1(x')}{M_2(x')}\, \mathrm{d}x'\right], \tag{1}$$

where C is a constant which need not be computed. Why is this solution unique?

ii. Transforming the Fokker–Planck equation

Assume now that M_2 is actually constant. Let $\mathsf{V}(x) \equiv \frac{1}{2}[M_1(x)]^2 + \frac{M_2}{2}\frac{\mathrm{d}M_1(x)}{\mathrm{d}x}$.

Perform the change of unknown function $p_{X,1}(t, x) = [p^{\text{st.}}_{X,1}(x)]^{1/2}\psi(t, x)$ in the Fokker–Planck equation, where $p^{\text{st.}}_{X,1}(x)$ is the stationary solution (1), and deduce the equation obeyed by $\psi(t, x)$. What do you recognize?

In the new language you just found, to which known problem is that of the Fokker–Planck equation for the Ornstein–Uhlenbeck process [$M_1(x) = \gamma x$, $M_2 = D$, $x \in \mathbb{R}$] equivalent?