Tutorial sheet 7

11. Characteristic functional of a stochastic process

In the lecture, the characteristic functional associated with a stochastic process $Y_X(t)$ has been defined as

$$G_{Y_X}[k(t)] \equiv \left\langle \exp\left[\mathrm{i}\int k(t)\,Y_X(t)\,\mathrm{d}t\right]\right\rangle,$$

with $k(t)$ a test function.

i. Characteristic functional and moments

Expand the exponential in power series of k and express the characteristic functional in terms of the n-time moments. How would you write the moment $\langle Y_X(t_1)Y_X(t_2)\cdots Y_X(t_n)\rangle$ as function of $G_{Y_X}[k(t)]$?

ii. Gaussian process

Give the characteristic functional associated with a Gaussian process.

12. Vibrating string

Consider a weightless elastic strings, whose extremities are fixed at points $x = 0$ and $x = L$ along some x-axis. Let $y(x)$ denote the displacement of the string transverse to this axis—for the sake of simplicity, we can assume that this displacement is one-dimensional—at position x. For small displacements, one can show that the elastic energy associated with a given profile $y(x)$ reads

$$E[y(x)] = \int_0^L \frac{1}{2}k\left[\frac{\mathrm{d}y(x)}{\mathrm{d}x}\right]^2\mathrm{d}x, \tag{1}$$

with k a positive constant.

When the string undergoes thermal fluctuations, induced by its environment at temperature T, $y(x)$ becomes a random function (of position, instead of time), where one expects that the probability for a given $y(x)$ should be proportional to $\mathrm{e}^{-\beta E[y(x)]}$ with $\beta = 1/k_BT$. Here, we wish to consider a discretized version of the problem and view $y(x)$ as the realization of a stochastic function $Y(x)$.

i. Let $n \in \mathbb{N}$. Consider n points $0 < x_1 < x_2 < \cdots < x_n < L$ and let y_j be the displacement of the string at point x_j. Write down the energy of the string, assuming that it is straight between two successive points x_j, x_{j+1}.

Hint: For the sake of brevity, one can introduce the notations $x_0 = 0$, $x_{n+1} = L$, $y_0 = y_{n+1} = 0$.

ii. We introduce the n-point probability density

$$p_n(x_1,y_1;\ldots;x_n,y_n) = \sqrt{\frac{2\pi L}{k\beta}}\prod_{j=0}^{n}\sqrt{\frac{k\beta}{2\pi(x_{j+1}-x_j)}}\,\exp\left[-\frac{k\beta}{2}\frac{(y_{j+1}-y_j)^2}{x_{j+1}-x_j}\right],$$

which for large n, agrees with the anticipated factor $\mathrm{e}^{-\beta E[y(x)]}$ (are you convinced of that?).

Show that the various p_n satisfy the 4 properties given in the lecture. Write down the single-point and two-point averages $\langle Y(x_1)\rangle$ and $\langle Y(x_1)Y(x_2)\rangle$, as well as the autocorrelation function. Which properties does the process possess?