Tutorial sheet 5

9. Collisionless Boltzmann equation in the presence of a vector potential

The purpose of this exercise is to show the equivalence in the presence of an external vector potential of the two forms of the collisionless Boltzmann equation given in the lecture, namely the version involving the particle-number density f in phase space

$$\frac{\partial f(t,\vec{r},\vec{p})}{\partial t} + \dot{\vec{r}} \cdot \vec{\nabla}_{\vec{r}} f(t,\vec{r},\vec{p}) + \dot{\vec{p}} \cdot \vec{\nabla}_{\vec{p}} f(t,\vec{r},\vec{p}) = 0, \tag{1a}$$

and the formulation in terms of particle-number density $f_{\vec{v}}$ in position-velocity space

$$\frac{\partial f_{\vec{v}}(t,\vec{r},\vec{v})}{\partial t} + \dot{\vec{r}} \cdot \vec{\nabla}_{\vec{r}} f_{\vec{v}}(t,\vec{r},\vec{v}) + \dot{\vec{v}} \cdot \vec{\nabla}_{\vec{v}} f_{\vec{v}}(t,\vec{r},\vec{v}) = 0. \tag{1b}$$

i. Hamiltonian formalism for a charged particle

As a preliminary, we review the Hamiltonian description of the motion of charged particles in external fields.

a) Consider a particle of mass m and electric charge q, evolving in presence of a scalar potential $\phi(t,\vec{r})$ and a vector potential $\vec{A}(t,\vec{r})$. Give the Hamilton function for the particle.

b) Write down the Hamilton evolution equation for the position $\vec{r}$ of the particle. Deduce the relation between the particle velocity $\vec{v}$ and its canonical momentum $\vec{p}$.

c) Recall the expression of the electric and magnetic fields $\vec{E}$, $\vec{B}$ as functions of the potentials, as well as of the Lorentz force on a charged particle.

d) Consider now the Hamilton evolution equation for the canonical momentum $\vec{p}$ of the particle. Check that it gives Newton's second law with the Lorentz force. What can you say about the time derivative $\dot{\vec{p}} \equiv \mathrm{d}\vec{p}/\mathrm{d}t$?

ii. Collisionless Boltzmann equation

a) Using the result obtained in question **i. b**, express $f_{\vec{v}}(t,\vec{r},\vec{v})$ as a function of $f(t,\vec{r},\vec{p})$ and the particle mass m.

From this relation, we shall now successively rewrite the three terms in the left-hand side of Eq. (1a) in terms of the derivatives of $f_{\vec{v}}(t,\vec{r},\vec{v})$.

b) Express $\dfrac{\partial f(t,\vec{r},\vec{p})}{\partial t}$ as a function of derivatives of $f_{\vec{v}}(t,\vec{r},\vec{v})$.

c) Express $\dfrac{\partial f(t,\vec{r},\vec{p})}{\partial x}$ as a function of derivatives of $f_{\vec{v}}(t,\vec{r},\vec{v})$. Deduce the expression of $\dot{\vec{r}} \cdot \vec{\nabla}_{\vec{r}} f(t,\vec{r},\vec{p})$.

Hint: Recognize $\Big[\big(\vec{v}\cdot\vec{\nabla}\big)\vec{A}(t,\vec{r})\Big] \cdot \vec{\nabla}_{\vec{v}} f_{\vec{v}}(t,\vec{r},\vec{v})$ in one of the terms that appear.

d) Express $\vec{\nabla}_{\vec{p}} f(t,\vec{r},\vec{p})$ as a function of derivatives of $f_{\vec{v}}(t,\vec{r},\vec{v})$. Use the expression of $\dot{\vec{p}}$ found in question **i. d** to conclude the computation of the left-hand side of Eq. (1a).