

Tutorial sheet 4

7. Two-particle phase space density in a classical ideal gas

The purpose of this exercise is to show that even in a classical ideal gas of identical particles, (small) correlations between particles arise when their total number N is fixed.

i. Canonical equilibrium

Consider first the case of a classical gas of N non-interacting identical particles, each of which is described by a Hamilton function h , so that the total Hamilton function reads

$$H_N = h(1) + h(2) + \dots + h(N),$$

where the positions and momenta of the particles have been for brevity denoted through the particle label. At thermodynamic equilibrium, the canonical partition function for this gas is

$$Z_N(\beta, \mathcal{V}) = \frac{[Z_1(\beta, \mathcal{V})]^N}{N!},$$

with Z_1 the partition function for a single particle—which need not be specified hereafter.

- a) Compute first the single-particle phase space density $f_1(\vec{r}_1, \vec{p}_1)$.
- b) Write down the two-particle phase space density $f_2(\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2)$, and compare it to the product of the single-particle densities for particles 1 and 2. What you notice?

ii. Grand canonical equilibrium

The number of particles in the gas is now allowed to vary, so that the proper description at thermodynamic equilibrium takes place in the grand canonical ensemble, with the “chemical” Lagrange multiplier $\alpha = \beta\mu$.

- a) Recall the expression of the grand canonical partition function $Z(\beta, \mathcal{V}, \alpha)$ as function of the canonical partition functions Z_N and α , then as function of Z_1 and α .
- b) Derive the single-particle phase space density $f_1(\vec{r}_1, \vec{p}_1)$.

Hint: Consider first the contribution to f_1 coming from the case when the gas consists of N particles, then carefully sum over all possible values of N .

- c) Compute the two-particle phase space density $f_2(\vec{r}_1, \vec{p}_1, \vec{r}_2, \vec{p}_2)$ and rewrite it as function of the single-particle densities for particles 1 and 2. Compare with the result in the canonical approach.

8. Collisionless evolution of the single-particle phase space density

Let $f(t, \vec{r}, \vec{p})$ denote the single-particle phase space density of a system of non-interacting particles with mass m evolving in the absence of long-range interactions deriving from a vector potential. Consider the particles which at time t are in an infinitesimal volume element $d^3\vec{r} d^3\vec{p}$ about point (\vec{r}, \vec{p}) .

Where are these particles at the instant $t + dt$? Show that the volume element $d^3\vec{r}' d^3\vec{p}'$ they then occupy equals $d^3\vec{r} d^3\vec{p}$. Derive the partial differential equation governing the evolution of f .