Tutorial sheet 1

1. Flow of a gas between two containers at different temperatures and pressures

Consider a composite system of two containers A and B with a classical ideal monoatomic gas of particles of mass m at temperatures T_A, T_B and pressures $\mathcal{P}_A$, $\mathcal{P}_B$ respectively. Let $\Delta T \equiv T_B - T_A$ and $\Delta\mathcal{P} \equiv \mathcal{P}_B - \mathcal{P}_A$ denote the temperature and pressure differences.

i. Recall the expression of the Maxwell–Boltzmann distribution $p(\vec{v})$ for the velocities in an ideal gas at temperature T. Assuming that the particle density is uniform, write down the number density $f(\vec{r},\vec{v})\,\mathrm{d}^3\vec{v}$ of particles per unit volume with a velocity between $\vec{v}$ and $\vec{v}+\mathrm{d}^3\vec{v}$.

ii. Particle flow

A small hole of cross section $\mathcal{S}$ in the wall separating the containers allows gas to slowly flow from one container to the other. Where are at time t the gas particles which will traverse the hole with a given velocity $\vec{v}$ between t and $t+\mathrm{d}t$? Show that the number of particles flowing from container A to container B per unit time is

$$\mathcal{J}_N^{(A)} = \frac{\mathcal{P}_A\mathcal{S}}{\sqrt{2\pi m k_B T_A}}.$$

Deduce the overall particle flux $\mathcal{J}_N \equiv \mathcal{J}_N^{(A)} - \mathcal{J}_N^{(B)}$ and express it as a function of ΔT and $\Delta\mathcal{P}$, assuming those differences are small.

iii. Energy flow

Show that the flow of (kinetic) energy through the hole from container A to container B per unit time is

$$\mathcal{J}_E^{(A)} = \sqrt{\frac{2k_BT_A}{\pi m}}\,\mathcal{P}_A\mathcal{S}.$$

Deduce the overall energy flux $\mathcal{J}_E \equiv \mathcal{J}_E^{(A)} - \mathcal{J}_E^{(B)}$ and express it as a function of ΔT and $\Delta\mathcal{P}$.

iv. The chemical potential of a classical ideal gas is given by

$$\mu(T,\mathcal{V},N) = -k_BT\ln\left[\frac{\mathcal{V}}{N}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\right].$$

Express $\mathcal{J}_N$ and $\mathcal{J}_E$ as a function of the differences (beware the signs!)

$$\Delta\left(\frac{1}{T}\right) \equiv \frac{1}{T_B} - \frac{1}{T_A} \quad\text{and}\quad \Delta\left(-\frac{\mu}{T}\right) \equiv \frac{\mu_A}{T_A} - \frac{\mu_B}{T_B}.$$

What do you recognize?

2. Energy fluctuations and heat capacity

The internal energy U of a system coupled to a heat reservoir, with which it can exchange energy, is a random variable. Give its variance as a function of the system heat capacity at constant volume.

Hint: Consider derivatives of the logarithm of the (canonical) partition function of equilibrium statistical mechanics.