# Tutorial sheet 9

**Discussion topic:** Dynamical similarity and the Reynolds number. You could also educate yourself on the topic of Life at low Reynolds number and the "scallop theorem" by reading E. M. Purcell's article (also accessible via the web page of the lectures)

## 25. Equations of fluid dynamics in a uniformly rotating reference frame

This exercise is inspired by Chapter 14.5.1 of *Modern Classical Physics* by Roger D. Blandford and Kip S. Thorne.

For the study of various physical problems (see examples in question **iv.a**), it may be more convenient to study the dynamics of a fluid from a reference frame $\mathcal{R}_{\Omega_0}$ in uniform rotation with angular velocity $\vec{\Omega}_0$ with respect to an inertial frame $\mathcal{R}_0$.
In exercise **12**, you already investigated hydrostatics in a rotating reference frame: in that case only the centrifugal acceleration plays a role, which can be entirely recast as the effect of a potential energy $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big(\vec{\Omega}_0 \times \vec{r}\big)^2$ leading to the centrifugal inertial force density $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$. The purpose of this exercise is to generalize that result to the derivation of (some of) the equations governing a flowing Newtonian fluid.

### i. Kinematics

Recall the expressions of the centrifugal and Coriolis accelerations acting on a small fluid element in terms of its position vector $\vec{r}$ and velocity $\vec{\mathsf{v}}$ (measured in $\mathcal{R}_{\Omega_0}$) and of the angular velocity.

### ii. Incompressibility condition

Writing down the relation between the velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}_{\Omega_0}$ and that measured in $\mathcal{R}_0$, show that the incompressibility condition valid in the inertial frame leads to $\vec{\nabla}\cdot\vec{\mathsf{v}} = 0$.

### iii. Navier–Stokes equation

Show that the incompressible Navier–Stokes equation from the point of view of an observer at rest in the rotating reference frame $\mathcal{R}_{\Omega_0}$ reads (the variables are omitted)

$$\frac{\mathrm{D}\vec{\mathsf{v}}}{\mathrm{D}t} = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}_{\text{eff.}} + \nu\triangle\vec{\mathsf{v}} - 2\vec{\Omega}_0 \times \vec{\mathsf{v}} \tag{1}$$

where $\mathcal{P}_{\text{eff.}} = \mathcal{P} + \rho\big(\Phi + \Phi_{\text{cen.}}\big)$, with $\Phi$ the potential energy from which (non-inertial) volume forces acting on the fluid derive. Check that you recover the equation of hydrostatics found in exercise **12**.

### iv. Dimensionless numbers and limiting cases

**a)** Let $L_c$ resp. $\mathsf{v}_c$ denote a characteristic length resp. velocity for a given flow. The Ekman and Rossby numbers are respectively defined as

$$\mathrm{Ek} \equiv \frac{\nu}{|\Omega_0|L_c^2} \qquad , \qquad \mathrm{Ro} \equiv \frac{\mathsf{v}_c}{|\Omega_0|L_c}.$$

Compute Ek and Ro in a few numerical examples:
– $L_c \approx 100$ km, $\mathsf{v}_c \approx 10\ \mathrm{m\cdot s^{-1}}$, $\Omega_0 \approx 10^{-4}\ \mathrm{rad\cdot s^{-1}}$, $\nu \approx 10^{-5}\ \mathrm{m^2\cdot s^{-1}}$ (wind in the Earth atmosphere);
– $L_c \approx 1000$ km, $\mathsf{v}_c \approx 0.1\ \mathrm{m\cdot s^{-1}}$, $\Omega_0 \approx 10^{-4}\ \mathrm{rad\cdot s^{-1}}$, $\nu \approx 10^{-6}\ \mathrm{m^2\cdot s^{-1}}$ (ocean stream);
– $L_c \approx 10$ cm, $\mathsf{v}_c \approx 1\ \mathrm{m\cdot s^{-1}}$, $\Omega_0 \approx 10\ \mathrm{rad\cdot s^{-1}}$, $\nu \approx 10^{-6}\ \mathrm{m^2\cdot s^{-1}}$ (coffee/tea in your cup).

**b)** Assuming stationarity, which term in Eq. (1) is negligible (against which) at small Ekman number? at small Rossby number?

Write down the simplified equation of motion valid when both $\mathrm{Ek} \ll 1$ and $\mathrm{Ro} \ll 1$ (to which of the above examples does this correspond?). How do the (effective) pressure gradient $\vec{\nabla}\mathcal{P}_{\text{eff.}}$ and flow velocity stand relative to each other?

## 26. Vortex dynamics in Newtonian fluids

**i.** Show that in a barotropic fluid with only conservative forces, the vorticity $\vec{\omega}$ is governed by

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} - \vec{\nabla}\times\left[\vec{\mathrm{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})\right] = \frac{\eta}{\rho(t,\vec{r})}\triangle\vec{\omega}(t,\vec{r}). \tag{2}$$

**ii. Diffusion of a rectilinear vortex**

Consider the incompressible flow (with constant uniform $\rho$) with at $t=0$ a rectilinear vortex

$$\vec{\omega}(t{=}0,\vec{r}) = \frac{\Gamma_0}{2\pi r}\delta(z)\,\vec{\mathrm{e}}_z \tag{3}$$

along the $z$-axis. The system geometry suggests the use of cylindrical coordinates $(r,\theta,z)$.

**a)** Assuming (why does this make sense?) that at $t>0$ the vorticity is still along the $z$-direction and only depends on the distance $r$ from the axis: $\vec{\omega}(t,\vec{r}) = \omega^z(t,r)\vec{\mathrm{e}}_z$, show that Eq. (2) simplifies to a (known) partial differential equation for $\omega^z$.

**b)** Can you solve this differential equation with the initial condition (3)?[1] You should find that at time $t$ the vorticity extends over a region of typical width $\sqrt{4\eta t/\rho}$.

**c)** Assuming you obtained $\omega^z(t,r)$ at the previous step, you can now compute the circulation of the velocity field around a circle of radius $R$ centered on the $z$-axis. You should find

$$\Gamma(t,R) = \Gamma_0\left[1-\mathrm{e}^{-\rho R^2/(4\eta t)}\right]. \tag{4}$$

Comment on this result (*Hint*: compare with the lecture of May 4th).

[1] One possibility is to remember the lecture of May 25th, in particular the discussion of heat diffusion.