Tutorial sheet 8

Discussion topic: What are the fundamental equations governing the dynamics of non-relativistic Newtonian fluids?

21. Flow due to an oscillating plane boundary

Consider a rigid infinitely extended plane boundary ($y = 0$) that oscillates in its own plane with a sinusoidal velocity $U\cos(\omega t)\,\vec{e}_x$. The region $y > 0$ is filled with an incompressible Newtonian fluid with uniform kinematic shear viscosity ν. We shall assume that volume forces on the fluid are negligible, that the pressure is uniform and remains constant in time, and that the fluid motion induced by the plane oscillations does not depend on the coordinates x, z.

i. Determine the flow velocity $\vec{\mathsf{v}}(t, y)$ and plot the resulting profile.

ii. What is the characteristic thickness of the fluid layer in the vicinity of the plane boundary that follows the oscillations? Comment on your result.

22. Flow of a Newtonian fluid down a constant slope

A layer of Newtonian fluid is flowing under the influence of gravity (acceleration g) down a slope inclined at an angle α from the horizontal. The fluid itself is assumed to have a constant thickness h, so that its free surface is a plane parallel to its bottom, and the flow is steady, laminar and incompressible. One further assumes that the pressure at the free surface of the fluid as well as "at the ends" at large $|x|$ is constant—i.e., the flow is entirely caused by gravity, not by a pressure gradient.
To fix notations, let x denote the direction along which the fluid flows, with the basis vector oriented downstream, and y be the direction perpendicular to x, oriented upwards.

i. Show that the flow velocity magnitude v and pressure $\mathcal{P}$ of the fluid obey the equations

$$\begin{cases} \dfrac{\partial \mathsf{v}}{\partial x} = 0 \\ \eta \triangle \mathsf{v} = -\rho g \sin\alpha \\ \dfrac{\partial \mathcal{P}}{\partial y} = -\rho g \cos\alpha, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} \mathsf{v} = 0 & \text{at } y = 0 \\ \dfrac{\partial \mathsf{v}}{\partial y} = 0 & \text{at } y = h \\ \mathcal{P} = \mathcal{P}_0 & \text{at } y = h. \end{cases} \tag{2}$$

Determine the pressure and then the velocity profile.

ii. Compute the rate of volume flow ("volumetric flux") across a surface $\mathcal{S}$ perpendicular to the x-direction.

23. Taylor–Couette flow. Measurement of shear viscosity

A Couette viscometer consists of an annular gap, filled with fluid, between two concentric cylinders with height L. The outer cylinder (radius R_2) rotates around the common axis with angular velocity Ω_2, while the inner cylinder (radius R_1) remains motionless. The motion of the fluid is assumed to be two-dimensional, laminar, incompressible, and steady.

Throughout this exercise, we use a system of cylinder coordinates (r, φ, z) with the physicists' usual convention, i.e. the corresponding basis vectors are are normalized to unity.

i. Check that the continuity equation leads to $\mathsf{v}^r = 0$, with v^r the radial component of the flow velocity.

ii. Prove that the Navier–Stokes equation lead to the equations

$$\frac{\mathsf{v}^\varphi(r)^2}{r} = \frac{1}{\rho}\frac{\partial \mathcal{P}(r)}{\partial r} \tag{3}$$

$$\frac{\partial^2 \mathsf{v}^\varphi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \mathsf{v}^\varphi(r)}{\partial r} - \frac{\mathsf{v}^\varphi(r)}{r^2} = 0. \tag{4}$$

What is the meaning of Eq. (3)? Solve Eq. (4) with the ansatz $\mathsf{v}^\varphi(r) = ar + \frac{b}{r}$.

iii. One can show (can you?) that the $r\varphi$-component of the stress tensor is given by

$$\sigma^{r\varphi} = \eta\left(\frac{1}{r}\frac{\partial \mathsf{v}^r}{\partial \varphi} + \frac{\partial \mathsf{v}^\varphi}{\partial r} - \frac{\mathsf{v}^\varphi}{r}\right).$$

Show that $\sigma^{r\varphi} = -\frac{2b\eta}{r^2}$, where b is the same coefficient as above.

iv. A torque $\mathcal{M}_z$ is measured at the surface of the inner cylinder. How can the shear viscosity η of the fluid be deduced from this measurement?
Numerical example: $R_1 = 10$ cm, $R_2 = 11$ cm, $L = 10$ cm, $\Omega_2 = 10$ rad·s^{-1} and $\mathcal{M}_z = 7{,}246 \cdot 10^{-3}$ N·m.

24. Dimensional consideration for viscous flows in a tube

Consider the motion of a given fluid in a cylindrical tube of length L and of circular cross section under the action of a difference $\Delta\mathcal{P}$ between the pressures at the two ends of the tube. The relation between the pressure drop per unit length $\Delta\mathcal{P}/L$ and the magnitude of the mean velocity $\langle\mathsf{v}\rangle$—defined as the average over a cross section of the tube—is given by

$$\frac{\Delta\mathcal{P}}{L} = C\langle\mathsf{v}\rangle^n,$$

with C a constant that depends on the fluid mass density ρ, on the kinematic shear viscosity ν, and on the radius a of the tube cross section. n is a number which depends on the type of flow: $n = 1$ if the flow is laminar (this is the Hagen–Poiseuille law seen in the lecture), while measurements in turbulent flows by Hagen (1854) resp. Reynolds (1883) have given $n = 1.75$ resp. $n = 1.722$.

Assuming that C is—up to a pure number—a product of powers of ρ, ν and a, determine the exponents of these power laws using dimensional arguments.