Tutorial sheet 6

Discussion topic: What is a potential flow? What are the corresponding equations of motion?

15. Two-dimensional potential flow. Teapot effect

Consider a steady two-dimensional potential flow with velocity $\vec{\mathsf{v}}(x,y)$, with (x,y) Cartesian coordinates. The associated complex velocity potential is denoted $\phi(z)$, where $z = x + \mathrm{i}y$.

i. Consider the complex potential $\phi(z) = Az^n$ with $A \in \mathbb{R}$ and $n \geq 1/2$. Show that this potential allows you to describe the flow velocity in the sector $\widehat{\mathcal{E}}$ delimited by two walls making an angle $\alpha = \pi/n$.

ii. What can you say about the flow velocity in the vicinity of the end-corner of the sector $\widehat{\mathcal{E}}$?

Hint: Distinguish the cases $\alpha < \pi$ and $\alpha > \pi$.

iii. Teapot effect

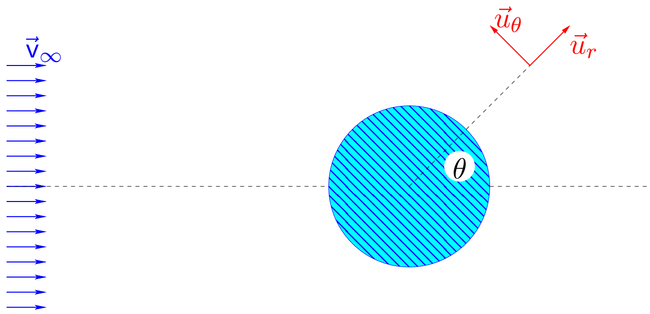
If one tries to pour tea "carefully" from a teapot, one will observe that the liquid will trickle along the lower side of the nozzle, instead of falling down into the cup waiting below. Explain this phenomenon using the flow profile introduced above (in the case $\alpha > \pi$) and the Bernoulli equation.

Literature: Jearl Walker, Scientific American, Oct. 1984 (= Spektrum der Wissenschaft, Feb. 1985).

iv. Assuming now that you are using the potential $\phi(z) = Az^n$ to model the flow of a river, which qualitative behavior can you anticipate for its bank?

16. Potential flow with a vortex. Magnus effect

The purpose of this exercise is to introduce a simplified model for the Magnus effect, which was discussed in the lectures.



One can show that the flow velocity of an incompressible perfect fluid around a cylinder of radius R at rest, with the uniform condition $\vec{\mathsf{v}}(\vec{r}) = \vec{\mathsf{v}}_\infty$ far from the cylinder—$\vec{\mathsf{v}}_\infty$ being perpendicular to the cylinder axis—, is given by

$$\vec{\mathsf{v}}(r,\theta) = \mathsf{v}_\infty \left[\left(1 - \frac{R^2}{r^2}\right) \cos\theta\, \vec{u}_r - \left(1 + \frac{R^2}{r^2}\right) \sin\theta\, \vec{u}_\theta \right], \tag{1}$$

where (r,θ) are polar coordinates—the third dimension (z), along the cylinder axis, plays no role—with the origin at the center of the cylinder (see Figure) and $\vec{u}_r$, $\vec{u}_\theta$ unit length vectors.

One superposes to the velocity field (1) a vortex with circulation Γ, corresponding to a flow velocity

$$\vec{\mathsf{v}}(r,\theta) = \frac{\Gamma}{2\pi r} \vec{u}_\theta. \tag{2}$$

i. Let $C \equiv \Gamma/(4\pi R \mathsf{v}_\infty)$. Determine the points with vanishing velocity for the flow resulting from superposing (1) and (2).

Hint: Distinguish the two cases $C < 1$ and $C > 1$.

ii. How do the streamlines look like in each case? Comment on the physical meaning of the result.

iii. Express the force per unit length $\mathrm{d}\vec{F}/\mathrm{d}z$ exerted on the cylinder by the flow (1)+(2) as function of Γ, v_∞ and the mass density ρ of the fluid.

17. Flow of a liquid in the vicinity of a gas bubble

We assume that the flow of the liquid is radial: $\vec{\mathsf{v}} = \mathsf{v}(t,r)\,\vec{\mathrm{e}}_r$, where the gas bubble is assumed to sit at $\vec{r} = \vec{0}$. Throughout the exercise, the effect of the liquid-gas surface tension—which gives rise to a difference in pressure between both sides of the liquid-gas interface—is neglected.

i. a) Show that the liquid's flow is irrotational. (*Hint*: one can avoid the computation of the curl!)

b) Assuming in addition that the flow is incompressible, derive the expression of $\mathsf{v}(t,r)$ in terms of the bubble radius $R(t)$ and its derivative $\dot{R}(t)$. Deduce therefrom the velocity potential.

ii. One assumes that the gas inside the bubble is an ideal gas which evolves adiabatically when the bubble radius varies, i.e. that its pressure—assumed to be uniform—and volume obey $\mathcal{P}\mathcal{V}^\gamma = \text{constant}$, where γ is the heat capacity ratio. Let $\mathcal{P}_0$ be the value of the pressure at infinity and R_0 the bubble radius when the gas pressure equals $\mathcal{P}_0$.

a) Neglecting the gas flow, give the expression of the pressure inside the bubble in terms of the radius.

b) Writing the Euler equation in terms of the velocity potential, show that $R(t)$ obeys the evolution equation

$$\ddot{R}(t)\,R(t) + \frac{3\big[\dot{R}(t)\big]^2}{2} = \frac{\mathcal{P}_0}{\rho}\left[\left(\frac{R_0}{R(t)}\right)^{3\gamma} - 1\right], \tag{3}$$

where ρ is the liquid mass density.

iii. Suppose now that the bubble radius slightly oscillates about the equilibrium value R_0. Writing $R(t) = R_0[1+\epsilon(t)]$ with $|\epsilon(t)| \ll 1$, derive the (linear!) evolution equation for $\epsilon(t)$. What is the frequency f of such small oscillations?

Numerical application: calculate f for air ($\gamma = 1.4$) bubbles with $R_0 = 1$ mm and $R_0 = 5$ mm in water ($\rho = 10^3$ kg/m^3) for $\mathcal{P}_0 = 10^5$ Pa.