Tutorial sheet 3

Discussion topics:

- What are the strain rate tensor, the rotation rate tensor, and the vorticity vector? How do they come about and what do they measure?
- What is the Reynolds transport theorem (and its utility)?

7. Two motions with cylindrical symmetry

In this exercise, we use a system of cylindrical coordinates (r, θ, z).

i. Pointlike source

Consider the fluid motion defined for $r \neq 0$ by the velocity field

$$\mathsf{v}^r(t,\vec{r}) = \frac{f(t)}{r}, \quad \mathsf{v}^\theta(t,\vec{r}) = 0, \quad \mathsf{v}^z(t,\vec{r}) = 0,$$

with f some scalar function.

a) Compute the volume expansion rate and the vorticity vector.

b) Mathematically, the velocity field is singular at $r = 0$. Thinking of the velocity profile, what do you have *physically* at that point if $f(t) > 0$? if $f(t) < 0$?

ii. Pointlike vortex

Consider now the fluid motion defined for $r \neq 0$ by the velocity field

$$\vec{\mathsf{v}}(t,\vec{r}) = \frac{\Gamma}{2\pi r}\vec{u}_\theta, \quad \Gamma \in \mathbb{R},$$

where $\vec{u}_\theta$ denotes a unit vector in the orthoradial direction.[1] Give the corresponding volume expansion rate and vorticity vector. Compute the *circulation* of the velocity field along a closed curve circling the z-axis. For which physical phenomenon could this motion be a (very crude!) model?

iii. The velocity fields of questions **i.** — assuming that $f(t)$ is time-independent — and **ii.** are analogous to the electrical or magnetic fields created by simple (stationary) distributions of electric charges or currents. Do you see which?

8. Symmetry of the stress tensor

Let $\boldsymbol{\sigma}_{ij} = -\mathbf{T}_{ij}$ denote the Cartesian components of the stress tensor in a continuous medium. Consider an infinitesimal cube of medium, whose edges (length $\mathrm{d}\ell$) are parallel to the axes of the coordinate system.

i. Explain why the k-component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the continuous medium obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\mathbf{T}_{ij}(\mathrm{d}\ell)^3$, with ϵ_{ijk} the usual Levi-Civita symbol.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the continuum mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?

[1] That is, $\vec{u}_\theta$ is in the plane perpendicular to the z-axis and orthogonal to the radial direction away from the z-axis.