

Tutorial sheet 2

Discussion topic: What are the Lagrangian and Eulerian descriptions? How is a fluid defined?

3. Stationary flow: second example

Consider the fluid flow whose velocity field $\vec{v}(t, \vec{r})$ has coordinates (in a given Cartesian system)

$$v^1(t, \vec{r}) = kx^2, \quad v^2(t, \vec{r}) = kx^1, \quad v^3(t, \vec{r}) = 0, \quad (1)$$

where k is a positive real number, while x^1, x^2, x^3 are the coordinates of the position vector \vec{r} .

- i. Determine the stream lines at an arbitrary instant t .
- ii. Let X^1, X^2, X^3 denote the coordinates of some arbitrary point M and let t_0 be the real number defined by

$$kt_0 = \begin{cases} -\text{Artanh}(X^2/X^1) & \text{if } |X^1| > |X^2| \\ 0 & \text{if } X^1 = \pm X^2 \\ -\text{Artanh}(X^1/X^2) & \text{if } |X^1| < |X^2|. \end{cases}$$

Write down a parameterization $x^1(t), x^2(t), x^3(t)$, in terms of a parameter denoted by t , of the coordinates of *the* stream line $\vec{x}(t)$ going through M such that $d\vec{x}(t)/dt$ at any point *equals* the velocity field at that point, and that either $x^1(t) = 0$ or $x^2(t) = 0$ for $t = t_0$.

iii. Viewing $\vec{x}(t)$ as the trajectory of a point—actually, of a fluid particle—, you already know the velocity of that point at time t (do you?). What is its acceleration $\vec{a}(t)$?

iv. Coming back to the velocity field (1), compute first its partial derivative $\partial\vec{v}(t, \vec{r})/\partial t$, then the *material derivative*

$$\frac{D\vec{v}(t, \vec{r})}{Dt} \equiv \frac{\partial\vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}]\vec{v}(t, \vec{r}).$$

Compare $\partial\vec{v}(t, \vec{r})/\partial t$ and $D\vec{v}(t, \vec{r})/Dt$ with the acceleration of a fluid particle found in question iii.

4. Lagrangian description: Jacobian determinant

Consider the twice continuously differentiable (\mathcal{C}^2) mapping $(t, \vec{R}) \mapsto \vec{r}(t, \vec{R})$ from “initial” position vectors at t_0 to those at time t . Let (X^1, X^2, X^3) resp. (x^1, x^2, x^3) denote the coordinates of \vec{R} resp. \vec{r} in some fixed system.

The *Jacobian determinant* $J(t, \vec{R})$ of the transformation $\vec{R} \mapsto \vec{r}$ is as usual the determinant of the matrix with elements $\partial x^i/\partial X^j$. Thanks to the hypotheses on the mapping $\vec{r}(t, \vec{R})$, this Jacobian has simple mathematical properties.

- i. Can you find a physical interpretation for $J(t, \vec{R})$? [*Hint*: Think of small volume elements.]
- ii. Using the initial value $J(t_0, \vec{R})$ in the reference configuration, as well as the invertibility and \mathcal{C}^2 -character of the mapping $\vec{r}(t, \vec{R})$, show that $J(t, \vec{R})$ is positive for $t \geq t_0$. What does this mean physically?
- iii. Consider the motion of a continuous medium defined for $t \geq 0$ by

$$x^1 = X^1 + ktX^2, \quad x^2 = X^2 + ktX^1, \quad x^3 = X^3,$$

where $k > 0$. One may for simplicity assume that the coordinates are Cartesian.

- a) Over which time range is this motion defined? [*Hint*: Jacobian determinant!]
- b) What are its pathlines?
- c) Determine the Eulerian description of this motion, i.e. the velocity field $\vec{v}(t, \vec{r})$.

5. Isotropy of pressure

Consider a geometrical point at position \vec{r} in a fluid at rest. The stress vector across every surface element going through this point is normal: $\vec{T}(\vec{r}) = -\mathcal{P}(\vec{r})\vec{e}_n$, with \vec{e}_n the unit vector orthogonal to the surface element under consideration. Show that the (hydrostatic) pressure \mathcal{P} is independent of the orientation of \vec{e}_n .

Hint: Consider the forces on the faces of an infinitesimal trirectangular tetrahedron.