Tutorial sheet 13

Discussion topic: What are the fundamental equations of the dynamics of a relativistic fluid? What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?

Hint: If the covariant derivatives d_μ in the following exercises upset you, choose Minkowski coordinates, in which $\mathrm{d}_\mu = \partial_\mu$.

34. Quantum number conservation

Consider a 4-current with components $N^\mu(x)$ obeying the continuity equation $\mathrm{d}_\mu N^\mu(\mathsf{x}) = 0$. Show that the quantity $\mathcal{N} = \int N^0(\mathsf{x})\,\mathrm{d}^3\vec{r}/c$ is a Lorentz scalar, by convincing yourself first that $\mathcal{N}$ can be rewritten in the form

$$\mathcal{N} = \frac{1}{c}\int_{x^0=\text{const.}} N^\mu(\mathsf{x})\,\mathrm{d}^3\sigma_\mu, \tag{1}$$

where $\mathrm{d}^3\sigma_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ is a 4-vector, with $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ the antisymmetric 4-tensor defined by

$$\mathrm{d}^3\mathcal{V}^{012} = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2, \quad \mathrm{d}^3\mathcal{V}^{021} = -\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^1, \quad \text{etc.}$$

and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric Levi–Civita tensor with the convention $\epsilon_{0123} = +1$, such that $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ represents the 3-dimensional hypersurface element in Minkowski space.

35. Energy-momentum tensor

Let $\mathcal{R}$ denote a fixed reference frame. Consider a perfect fluid whose local rest frame at a point x moves with velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}$. Show with the help of a Lorentz transformation that the Minkowski components of the energy-momentum tensor of the fluid at x are given to order $\mathcal{O}(|\vec{\mathsf{v}}|/c)$ by

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & \mathcal{P} & 0 & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & 0 & \mathcal{P} & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} & 0 & 0 & \mathcal{P} \end{pmatrix},$$

where for the sake of brevity the x-dependence of the various fields is omitted. Check the compatibility of this result with the general formula for $T^{\mu\nu}$ given in the lecture.

36. Equations of motion of a perfect relativistic fluid

In this exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

i. Check that the tensor with components $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ defines a projector on the subspace orthogonal to the 4-velocity.
Denoting by d_μ the components of the (covariant) 4-gradient, we define $\nabla^\nu \equiv \Delta^{\mu\nu}\mathrm{d}_\mu$. Can you see the rationale behind this notation?

ii. Show that the energy-momentum conservation equation for a perfect fluid is equivalent to the two equations

$$u^\mu \mathrm{d}_\mu\epsilon + (\epsilon+\mathcal{P})\mathrm{d}_\mu u^\mu = 0 \quad \text{and} \quad (\epsilon+\mathcal{P})u^\mu\mathrm{d}_\mu u^\nu + \nabla^\nu\mathcal{P} = 0. \tag{2}$$

Which known equation does the second one evoke?

37. A family of solutions of the dynamical equations for perfect relativistic fluids

Let $\{x^\mu\}$ denote Minkowski coordinates and $\tau^2 \equiv -x^\mu x_\mu$, where the "mostly plus" metric is used. Show that the following four-velocity, pressure and charge density constitute a solution of the equations describing the motion of a perfect relativistic fluid with equation of state $\mathcal{P} = K\varepsilon$ and a single conserved charge:

$$u^\mu(\mathsf{x}) = \frac{x^\mu}{\tau} \quad , \quad \mathcal{P}(\mathsf{x}) = \mathcal{P}_0 \left(\frac{\tau_0}{\tau}\right)^{3(1+K)} \quad , \quad n(\mathsf{x}) = n_0 \left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}\big(\sigma(\mathsf{x})\big), \tag{3}$$

with τ_0, $\mathcal{P}_0$, n_0 arbitrary constants and $\mathcal{N}$ an arbitrary function of a single argument, while σ is a function of spacetime coordinates with vanishing comoving derivative: $u^\mu \partial_\mu \sigma(\mathsf{x}) = 0$.