Tutorial sheet 12

Discussion topic: Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects play a role?

For the sake of brevity, throughout this exercise sheet the dependence of the various fields on the space and time variables is not written.

32. Thermal convection between two vertical plates

Consider a fluid in a gravitational potential $-\vec{\nabla}\Phi = \vec{g} \equiv g\,\vec{e}_z$, contained between two infinite vertical plates at $x = \pm d/2$. When the plates have the same uniform temperature, there exist a static "isothermal" solution of the equations of motion describing the fluid, in which the latter is at the same temperature $T_{\rm eq}$ everywhere.

Assume that the plate at $x = -d/2$ resp. $x = +d/2$ is at a uniform temperature T_- resp. T_+ with $T_- < T_+$: this will induce a motion of the fluid, which we want to investigate. For simplicity, we shall assume that the motion is steady, and that it constitutes a small perturbation of the equilibrium state in which both temperatures are equal. Accordingly, the pressure, temperature and mass density are written in the form

$$\mathcal{P} = \mathcal{P}_{\rm eq} + \delta\mathcal{P}\ ,\ T = T_{\rm eq} + \delta T\ ,\ \rho = \rho_{\rm eq} + \delta\rho, \tag{1}$$

where the quantities with the subscript eq. refer to the equilibrium state, which need not be further specified.

i. Show that the relevant equations (IX.8), (IX.9), (IX.12), (IX.13) of the lecture notes lead for the small quantities $\delta\mathcal{P}$, δT, $\delta\rho$ and $\vec{\mathsf{v}}$ to the system

$$\vec{\nabla}\cdot\vec{\mathsf{v}} = 0 \tag{2a}$$

$$\vec{\nabla}\big(\delta\mathcal{P}\big) = \delta\rho\,\vec{g} + \nu\rho_{\rm eq}\triangle\vec{\mathsf{v}} \tag{2b}$$

$$\vec{\mathsf{v}}\cdot\vec{\nabla}T_{\rm eq} = \alpha\triangle\big(\delta T\big) \tag{2c}$$

$$\delta\rho = -\alpha_{(\mathcal{V})}\rho_{\rm eq}\,\delta T \tag{2d}$$

where the stationarity assumption has already been used. How did you implement the assumed smallness of the "perturbations" to the static state? How can you already simplify Eq. (2c)?

ii. Let us assume that the new flow only depends on the x-coordinate, and that the y-direction plays no role at all; in particular, there is no component v_y. Let us further assume that the net mass flow through any plane $z =$ const. vanishes, i.e.

$$\int_{-d/2}^{d/2} \rho_{\rm eq}\,\mathsf{v}_z(x,y,z)\,{\rm d}x = 0 \tag{3}$$

for all y, z: this condition allows us to fully specify the "boundary" conditions obeyed by the velocity field.

a) Determine first the temperature-variation profile $\delta T(x)$ and deduce from it the mass density perturbation $\delta\rho(x)$. (*Hint*: Eqs. (2c)–(2d)).

b) Determine the velocity profile between the two plates. How do the streamlines look like?

iii. Time for some physics: what is absurd with the assumption of an infinite extent in the z-direction? Is there really heat convection in the flow determined in question **ii.**? Can you think of an (everyday-life) example—with finite plates!—corresponding to the setup considered here?

33. (1+1)-dimensional relativistic motion

On June 29th, the flow velocities considered in the lectures will reach the relativistic regime. To prepare for this event, you may refresh your knowledge on Special Relativity. This exercise is here to help you in that direction, and also introduces coordinates which will be used later in the lectures.

Consider a (1+1)-dimensional relativistic motion along a direction denoted as z, where the denomination “1+1” stands for one time and one spatial dimension. Throughout the exercise, the other two spatial directions play no role and the corresponding variables x, y are totally omitted. In addition, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein’s summation convention over repeated indices.

To describe the physics, one may naturally use Minkowski coordinates $(x^0, x^3) = (t, z)$, with corresponding derivatives $(\partial_0, \partial_3) = (\partial/\partial t, \partial/\partial z)$. If there is a high-velocity motion in the z-direction, a better choice might be to use the *proper time* τ and *spatial rapidity* ς such that[1]

$$x^{0'} \equiv \tau \equiv \sqrt{t^2 - z^2}, \quad x^{3'} \equiv \varsigma \equiv \frac{1}{2}\log\frac{t+z}{t-z} \quad \text{where } |z| \leq t. \tag{4}$$

The partial derivatives with respect to these new coordinates will be denoted $(\partial_{0'}, \partial_{3'}) = (\partial/\partial\tau, \partial/\partial\varsigma)$.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau\cosh\varsigma, \quad z = \tau\sinh\varsigma. \tag{5}$$

(*Hint:* Recognize $\frac{1}{2}\log\frac{1+u}{1-u}$).

ii. In a change of coordinates $\{x^\mu\} \to \{x^{\mu'}\}$, the contravariant components V^μ of a 4-vector transform according to $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$ (with summation over ν!) where $\Lambda^{\mu'}{}_\nu \equiv \partial x^{\mu'}/\partial x^\nu$.

Compute first from Eq. (5) the matrix elements $\Lambda^\nu{}_{\mu'} \equiv \partial x^\nu/\partial x^{\mu'}$ (with $\nu \in \{0,3\}$, $\mu' \in \{0',3'\}$) of the inverse transformation $\{V^{\mu'}\} \to \{V^\mu\}$. Inverting the 2×2-matrix you thus found, deduce the following relationship between the components of the 4-vector in the two coordinate systems

$$\begin{cases} V^{0'} = \cosh\varsigma\, V^0 - \sinh\varsigma\, V^3 \\ V^{3'} = -\dfrac{1}{\tau}\sinh\varsigma\, V^0 + \dfrac{1}{\tau}\cosh\varsigma\, V^3. \end{cases} \tag{6}$$

iii. Using the relation $\partial_\nu = \Lambda^{\mu'}{}_\nu \partial_{\mu'}$ and the matrix elements $\{\Lambda^{\mu'}{}_\nu\}$ you found in **ii.**—and which can be read off Eq. (6)—, express the “4-divergence” $\partial_\nu V^\nu$ of a 4-vector field V^ν in terms of the partial derivatives $\partial_{\mu'}$ and the components $V^{\mu'}$ in the (τ,ς)-system.

You should find a result that does not equal $\partial_{\mu'}V^{\mu'} = \partial_\tau V^\tau + \partial_\varsigma V^\varsigma$, which is why in the lecture notes the notation $\mathrm{d}_{\mu'}V^{\mu'}$ is used for the 4-divergence in an arbitrary coordinate system.

iv. Draw on a spacetime diagram—with t on the vertical axis and z on the horizontal axis—the lines of constant τ and those of constant ς.

Remark: The coordinates (τ,ς) are sometimes called *Milne coordinates*.

[1] ς = `\varsigma` is the word-final form for the lower case sigma, not to be confused with ζ (zeta).