Tutorial sheet 10

27. Flows at small Reynolds number

Check that an alternative form of the Stokes equation for creeping incompressible flows is[1]

$$\sum_{j=1}^{3} \frac{\partial \sigma^{ij}}{\partial x^j} = 0 \quad \forall i \in \{1,2,3\} \tag{1}$$

where σ^{ij} denotes the components (in a system of Cartesian coordinates) of the stress tensor — whose expression you can find in § III.3.3.b of the lecture notes.

28. Instability of the viscous Burgers equation

Neglecting the pressure term in the Navier–Stokes equation for a one-dimensional incompressible problem without external force yields the so-called *viscous Burgers equation*[2]

$$\frac{\partial \mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x} = \nu \frac{\partial^2 \mathsf{v}(t,x)}{\partial x^2}, \tag{2}$$

where $\nu \equiv \eta/\rho$ is the kinematic shear viscosity of the fluid. A trivial solution to this equation of motion is the steady uniform flow $\mathsf{v}(t,x) = \mathsf{v}_0$.

Let us add a perturbation $\delta\mathsf{v}(t,x)$.

a) Write down the linearized equation of motion governing the evolution of $\delta\mathsf{v}$ and derive the corresponding dispersion relation using an appropriate Fourier ansatz.

b) Fixing first $k \in \mathbb{R}$, check that the perturbation is exponentially damped in time.

c) Consider now a fixed $\omega \in \mathbb{R}$. How does the perturbation propagate along the x-direction? (*Hint*: For the sake of simplicity you may restrict your discussion to the small-viscosity case $\omega\nu \ll \mathsf{v}_0^2$.)

29. Instabilities in parallel shear flows

In the lectures we considered a number of simple steady incompressible flows with velocity of the form $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(y)\vec{\mathrm{e}}_x$, where x, y, z are Cartesian coordinates. For the stability of such so-called "parallel shear flows" there exist a number of results, some of which are discussed in this exercise. Throughout we assume that the mass density ρ_0 remains uniform and constant, and that there are no external forces.

i. Starting from the continuity and incompressible Navier–Stokes equations, write down the linearized equations of motion governing the evolution of perturbations $\delta\vec{\mathsf{v}}(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ of steady fields $\vec{\mathsf{v}}_0(\vec{r})$ and $\mathcal{P}_0(\vec{r})$, assuming $\vec{\mathsf{v}}_0(\vec{r}) = \mathsf{v}_0(y)\vec{\mathrm{e}}_x$.

One can show (Squire's theorem) that it is sufficient to investigate perturbations that are two-dimensional, i.e. that do not depend on z and such that $\delta\vec{\mathsf{v}}$ lies in the (x,y)-plane. To describe the latter, one can introduce the associated stream function $\psi(t,\vec{r})$, such that the non-zero components of $\delta\vec{\mathsf{v}}$ are given by $\delta\mathsf{v}^x = -\partial\psi/\partial y$ and $\delta\mathsf{v}^y = \partial\psi/\partial x$.

ii. Assume first that the fluid is perfect.

a) Using the linearized equations of motion you obtained in **i.**, show that the stream function satisfies the partial differential equation

$$\left[\frac{\partial}{\partial t} + \mathsf{v}_0(y)\frac{\partial}{\partial x}\right]\triangle\psi(t,\vec{r}) - \frac{\partial^2 \mathsf{v}_0(y)}{\partial y^2}\frac{\partial \psi(t,\vec{r})}{\partial x} = 0. \tag{3}$$

[1] A shorter (and thus more elegant?) form is $\vec{\nabla}\cdot\boldsymbol{\sigma} = \vec{0}$.

[2] You already encountered its inviscid version in exercise **19.**

b) Making the Fourier ansatz $\psi(t,\vec{r}) = \widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}$, show that Eq. (3) leads to *Rayleigh's equation*

$$\big[\mathsf{v}_0(y) - c(k)\big]\left(\frac{\partial^2}{\partial y^2} - k^2\right)\widetilde{\psi}(y) - \frac{\partial^2 \mathsf{v}_0(y)}{\partial y^2}\widetilde{\psi}(y) = 0, \tag{4}$$

where $c(k) \equiv \omega/k$.

For a given profile $\mathsf{v}_0(y)$ of the unperturbed flow and a fixed wavenumber k, this is an eigenvalue equation, whose solutions are eigenfunctions $\widetilde{\psi}(y)$ with associated eigenvalues $c(k)$. Show that if $\widetilde{\psi}$ is an eigenfunction associated with some eigenvalue $c(k)$, then its complex conjugate $\widetilde{\psi}^*$ is also an eigenfunction, with eigenvalue $c(k)^*$. What does this mean for the stability of the unperturbed flow in case one of the eigenvalues is not real?

iii. If you still have time, you may show that in a Newtonian incompressible fluid, Rayleigh's equation is replaced by the *Orr–Sommerfeld equation*

$$\big[\mathsf{v}_0(y) - c(k)\big]\left(\frac{\partial^2}{\partial y^2} - k^2\right)\widetilde{\psi}(y) - \frac{\partial^2 \mathsf{v}_0(y)}{\partial y^2}\widetilde{\psi}(y) = \frac{\nu}{\mathrm{i}k}\left(\frac{\partial^2}{\partial y^2} - k^2\right)^2\widetilde{\psi}(y), \tag{5}$$

with $\nu \equiv \eta/\rho_0$ the kinematic shear viscosity of the fluid.