

Tutorial sheet 1

Discussion topic: Which idealizations underlie the description of a macroscopic many-body system as a continuous medium? How is local thermodynamic equilibrium defined?

1. Wave equation

Consider a scalar field $\phi(t, x)$ which obeys the partial differential equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \phi(t, x) = 0 \quad (1)$$

with initial conditions $\phi(0, x) = e^{-x^2}$, $\partial_t \phi(0, x) = 0$. Determine the solution $\phi(t, x)$ for $t > 0$.

2. Stationary flow: first example

(This exercise introduces a number of concepts which will only be introduced in later lectures; this should pose you no difficulty.)

Consider the stationary flow defined in the region $x^1 > 0$, $x^2 > 0$ by its velocity field

$$\vec{v}(t, \vec{r}) = k(-x^1 \vec{e}_1 + x^2 \vec{e}_2) \quad (2)$$

with k a positive constant, $\{\vec{e}_i\}$ the basis vectors of a Cartesian coordinate system and $\{x^i\}$ the coordinates of the position vector \vec{r} .

i. Vector analysis

a) Compute the divergence $\vec{\nabla} \cdot \vec{v}(t, \vec{r})$ of the velocity field (2). Check that your result is consistent with the existence of a scalar function $\psi(t, \vec{r})$ (the *stream function*) such that

$$\vec{v}(t, \vec{r}) = -\vec{\nabla} \times [\psi(t, \vec{r}) \vec{e}_3] \quad (3)$$

and determine $\psi(t, \vec{r})$ — there is an arbitrary additive constant, which you may set equal to zero. What are the lines of constant $\psi(t, \vec{r})$?

b) Compute now the curl $\vec{\nabla} \times \vec{v}(t, \vec{r})$ and deduce therefrom the existence of a scalar function $\varphi(t, \vec{r})$ (the *velocity potential*) such that

$$\vec{v}(t, \vec{r}) = -\vec{\nabla} \varphi(t, \vec{r}). \quad (4)$$

(*Hint:* remember a theorem you saw in your lectures on classical mechanics and/or electromagnetism.) What are the lines of constant $\varphi(t, \vec{r})$?

ii. Stream lines

Determine the *stream lines* at some arbitrary time t . The latter are by definition lines $\vec{\xi}(\lambda)$ whose tangent is everywhere parallel to the instantaneous velocity field, with λ a parameter along the stream line. That is, they obey the condition

$$\frac{d\vec{\xi}(\lambda)}{d\lambda} = \alpha(\lambda) \vec{v}(t, \vec{\xi}(\lambda))$$

with $\alpha(\lambda)$ a scalar function, or equivalently

$$\frac{d\xi^1(\lambda)}{v^1(t, \vec{\xi}(\lambda))} = \frac{d\xi^2(\lambda)}{v^2(t, \vec{\xi}(\lambda))} = \frac{d\xi^3(\lambda)}{v^3(t, \vec{\xi}(\lambda))},$$

with $d\xi^i(\lambda)$ the coordinates of the (infinitesimal) tangent vector to the stream line.