Tutorial sheet 1

Discussion topic: Which idealizations underlie the description of a macroscopic many-body system as a continuous medium? How is local thermodynamic equilibrium defined?

1. Wave equation

Consider a scalar field $\phi(t,x)$ which obeys the partial differential equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi(t,x) = 0 \tag{1}$$

with initial conditions $\phi(0,x) = \mathrm{e}^{-x^2}$, $\partial_t\phi(0,x) = 0$. Determine the solution $\phi(t,x)$ for $t > 0$.

2. Stationary flow: first example

(This exercise introduces a number of concepts which will only be introduced in later lectures; this should pose you no difficulty.)

Consider the stationary flow defined in the region $x^1 > 0$, $x^2 > 0$ by its velocity field

$$\vec{\mathsf{v}}(t,\vec{r}) = k(-x^1\,\vec{\mathrm{e}}_1 + x^2\,\vec{\mathrm{e}}_2) \tag{2}$$

with k a positive constant, $\{\vec{\mathrm{e}}_i\}$ the basis vectors of a Cartesian coordinate system and $\{x^i\}$ the coordinates of the position vector $\vec{r}$.

i. Vector analysis

a) Compute the divergence $\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})$ of the velocity field (2). Check that your result is consistent with the existence of a scalar function $\psi(t,\vec{r})$ (the *stream function*) such that

$$\vec{\mathsf{v}}(t,\vec{r}) = -\vec{\nabla}\times\left[\psi(t,\vec{r})\,\vec{\mathrm{e}}_3\right] \tag{3}$$

and determine $\psi(t,\vec{r})$ — there is an arbitrary additive constant, which you may set equal to zero. What are the lines of constant $\psi(t,\vec{r})$?

b) Compute now the curl $\vec{\nabla}\times\vec{\mathsf{v}}(t,\vec{r})$ and deduce therefrom the existence of a scalar function $\varphi(t,\vec{r})$ (the *velocity potential*) such that

$$\vec{\mathsf{v}}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r}). \tag{4}$$

(*Hint*: remember a theorem you saw in your lectures on classical mechanics and/or electromagnetism.) What are the lines of constant $\varphi(t,\vec{r})$?

ii. Stream lines

Determine the *stream lines* at some arbitrary time t. The latter are by definition lines $\vec{\xi}(\lambda)$ whose tangent is everywhere parallel to the instantaneous velocity field, with λ a parameter along the stream line. That is, they obey the condition

$$\frac{\mathrm{d}\vec{\xi}(\lambda)}{\mathrm{d}\lambda} = \alpha(\lambda)\,\vec{\mathsf{v}}(t,\vec{\xi}(\lambda))$$

with $\alpha(\lambda)$ a scalar function, or equivalently

$$\frac{\mathrm{d}\xi^1(\lambda)}{\mathsf{v}^1(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^2(\lambda)}{\mathsf{v}^2(t,\vec{\xi}(\lambda))} = \frac{\mathrm{d}\xi^3(\lambda)}{\mathsf{v}^3(t,\vec{\xi}(\lambda))},$$

with $\mathrm{d}\xi^i(\lambda)$ the coordinates of the (infinitesimal) tangent vector to the stream line.

Tutorial sheet 2

Discussion topic: What are the Lagrangian and Eulerian descriptions? How is a fluid defined?

3. Stationary flow: second example

Consider the fluid flow whose velocity field $\vec{\mathsf{v}}(t,\vec{r})$ has coordinates (in a given Cartesian system)

$$\mathsf{v}^1(t,\vec{r}) = kx^2, \quad \mathsf{v}^2(t,\vec{r}) = kx^1, \quad \mathsf{v}^3(t,\vec{r}) = 0, \tag{1}$$

where k is a positive real number, while x^1, x^2, x^3 are the coordinates of the position vector $\vec{r}$.

i. Determine the stream lines at an arbitrary instant t.

ii. Let X^1, X^2, X^3 denote the coordinates of some arbitrary point M and let t_0 be the real number defined by

$$kt_0 = \begin{cases} -\mathrm{Artanh}(X^2/X^1) & \text{if } |X^1| > |X^2| \\ 0 & \text{if } \ X^1 = \pm X^2 \\ -\mathrm{Artanh}(X^1/X^2) & \text{if } |X^1| < |X^2|. \end{cases}$$

Write down a parameterization $x^1(t)$, $x^2(t)$, $x^3(t)$, in terms of a parameter denoted by t, of the coordinates of *the* stream line $\vec{x}(t)$ going through M such that $\mathrm{d}\vec{x}(t)/\mathrm{d}t$ at any point *equals* the velocity field at that point, and that either $x^1(t) = 0$ or $x^2(t) = 0$ for $t = t_0$.

iii. Viewing $\vec{x}(t)$ as the trajectory of a point—actually, of a fluid particle—, you already know the velocity of that point at time t (do you?). What is its acceleration $\vec{a}(t)$?

iv. Coming back to the velocity field (1), compute first its partial derivative $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$, then the *material derivative*

$$\frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t} \equiv \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}).$$

Compare $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$ and $\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})/\mathrm{D}t$ with the acceleration of a fluid particle found in question **iii.**

4. Lagrangian description: Jacobian determinant

Consider the twice continuously differentiable ($\mathscr{C}^2$) mapping $(t,\vec{R}) \mapsto \vec{r}(t,\vec{R})$ from "initial" position vectors at t_0 to those at time t. Let (X^1, X^2, X^3) resp. (x^1, x^2, x^3) denote the coordinates of $\vec{R}$ resp. $\vec{r}$ in some fixed system.

The *Jacobian determinant* $J(t,\vec{R})$ of the transformation $\vec{R} \mapsto \vec{r}$ is as usual the determinant of the matrix with elements $\partial x^i/\partial X^j$. Thanks to the hypotheses on the mapping $\vec{r}(t,\vec{R})$, this Jacobian has simple mathematical properties.

i. Can you find a physical interpretation for $J(t,\vec{R})$? [*Hint*: Think of small volume elements.]

ii. Using the initial value $J(t_0,\vec{R})$ in the reference configuration, as well as the invertibility and $\mathscr{C}^2$-character of the mapping $\vec{r}(t,\vec{R})$, show that $J(t,\vec{R})$ is positive for $t \geq t_0$. What does this mean physically?

iii. Consider the motion of a continuous medium defined for $t \geq 0$ by

$$x^1 = X^1 + ktX^2, \quad x^2 = X^2 + ktX^1, \quad x^3 = X^3,$$

where $k > 0$. One may for simplicity assume that the coordinates are Cartesian.

a) Over which time range is this motion defined? [*Hint*: Jacobian determinant!]

b) What are its pathlines?

c) Determine the Eulerian description of this motion, i.e. the velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

5. Isotropy of pressure

Consider a geometrical point at position $\vec{r}$ in a fluid at rest. The stress vector across every surface element going through this point is normal: $\vec{T}(\vec{r}) = -\mathcal{P}(\vec{r})\,\vec{e}_n$, with $\vec{e}_n$ the unit vector orthogonal to the surface element under consideration. Show that the (hydrostatic) pressure $\mathcal{P}$ is independent of the orientation of $\vec{e}_n$.

Hint: Consider the forces on the faces of an infinitesimal trirectangular tetrahedron.

Tutorial sheet 2 (supplement)

6. Yet another example of motion of a deformable continuous medium

Consider the motion defined in a system of Cartesian coordinates with basis vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ by the velocity field with components

$$\mathsf{v}^1(t,\vec{r}) = f_1(t,x^2), \quad \mathsf{v}^2(t,\vec{r}) = f_2(t,x^1), \quad \mathsf{v}^3(t,\vec{r}) = 0,$$

with f_1, f_2 two continuously differentiable functions.

Compute the strain rate tensor $\mathbf{D}(t,\vec{r})$ for this motion. What is the volume expansion rate? Give the rotation rate tensor $\mathbf{R}(t,\vec{r})$ and the vorticity vector. Under which condition(s) on the functions f_1, f_2 does the motion become irrotational?

Tutorial sheet 3

Discussion topics:

- What are the strain rate tensor, the rotation rate tensor, and the vorticity vector? How do they come about and what do they measure?
- What is the Reynolds transport theorem (and its utility)?

7. Two motions with cylindrical symmetry

In this exercise, we use a system of cylindrical coordinates (r, θ, z).

i. Pointlike source

Consider the fluid motion defined for $r \neq 0$ by the velocity field

$$\mathsf{v}^r(t, \vec{r}) = \frac{f(t)}{r}, \quad \mathsf{v}^\theta(t, \vec{r}) = 0, \quad \mathsf{v}^z(t, \vec{r}) = 0,$$

with f some scalar function.

a) Compute the volume expansion rate and the vorticity vector.

b) Mathematically, the velocity field is singular at $r = 0$. Thinking of the velocity profile, what do you have *physically* at that point if $f(t) > 0$? if $f(t) < 0$?

ii. Pointlike vortex

Consider now the fluid motion defined for $r \neq 0$ by the velocity field

$$\vec{\mathsf{v}}(t, \vec{r}) = \frac{\Gamma}{2\pi r}\vec{u}_\theta, \quad \Gamma \in \mathbb{R},$$

where $\vec{u}_\theta$ denotes a unit vector in the orthoradial direction.[1] Give the corresponding volume expansion rate and vorticity vector. Compute the *circulation* of the velocity field along a closed curve circling the z-axis. For which physical phenomenon could this motion be a (very crude!) model?

iii. The velocity fields of questions **i.** — assuming that $f(t)$ is time-independent — and **ii.** are analogous to the electrical or magnetic fields created by simple (stationary) distributions of electric charges or currents. Do you see which?

8. Symmetry of the stress tensor

Let $\boldsymbol{\sigma}_{ij} = -\mathbf{T}_{ij}$ denote the Cartesian components of the stress tensor in a continuous medium. Consider an infinitesimal cube of medium, whose edges (length $\mathrm{d}\ell$) are parallel to the axes of the coordinate system.

i. Explain why the k-component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the continuous medium obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\mathbf{T}_{ij}(\mathrm{d}\ell)^3$, with ϵ_{ijk} the usual Levi-Civita symbol.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the continuum mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?

[1] That is, $\vec{u}_\theta$ is in the plane perpendicular to the z-axis and orthogonal to the radial direction away from the z-axis.

Tutorial sheet 4

Discussion topics:

- Give the basic equations governing the dynamics of perfect fluids.
- What is the Bernoulli equation? Give some examples of application.

9. Simplified model of star

In an oversimplified approach, one may model a star as a sphere of fluid—a plasma—with uniform mass density ρ. This fluid is in mechanical equilibrium under the influence of pressure $\mathcal{P}$ and gravity. Throughout this exercise, the rotation of the star is neglected.

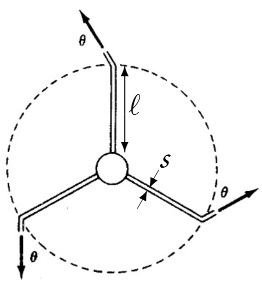
i. Determine the gravitational field at a distance r from the center of the star.

ii. Assuming that the pressure only depend on r, write down the equation expressing the mechanical equilibrium of the fluid. Determine the resulting function $\mathcal{P}(r)$. Compute the pressure at the star center as function of the mass M and radius R of the star. Calculate the numerical value of this pressure for $M = 2 \times 10^{30}$ kg (solar mass) and $R = 7 \times 10^{8}$ m (solar radius).

iii. The matter constituting the star is assumed to be an electrically neutral mixture of hydrogen nuclei and electrons. Show that the order of magnitude of the total particle number density of that plasma is $n \approx 2\rho/m_p$, with m_p the proton mass. Estimate the temperature at the center of the sun.
Hint: $m_p = 1.6 \times 10^{-27}$ kg; $k_B = 1.38 \times 10^{-23}$ J $\cdot$ K^{-1}.

10. Water sprinkler

The horizontal lawn sprinkler schematized below is fed water through its center with a mass flow rate Q. Assuming that water is a perfect incompressible fluid, determine the steady rotation rate as function of Q, the cross section area s of the pipes, their length ℓ, and the angle θ of the emerging water jets with respect to the respective pipes.



11. Rotating fluid in a uniform gravitational potential

Consider a perfect fluid contained in a straight cylindrical vessel which rotates with constant angular velocity $\vec{\Omega} = \Omega\,\vec{e}_3$ about its vertical axis, the whole system being placed in a uniform gravitational field $-g\,\vec{e}_3$. Assuming that the fluid rotates with the same angular velocity and that its motion is incompressible, determine the shape of the free surface of the fluid.

Hint: Despite the geometry, working with Cartesian coordinates is quite straightforward. At the free surface, the fluid pressure is constant (it equals the atmospheric pressure).

Tutorial sheet 5

Discussion topic: What is Kelvin's circulation theorem? What does it imply for the vorticity?

12. Statics of rotating fluids

This exercise is strongly inspired by Chapter 13.3.3 of *Modern Classical Physics* by Roger D. Blandford and Kip S. Thorne.

Consider a fluid, bound by gravity, which is rotating rigidly, i.e. with a uniform angular velocity $\vec{\Omega}_0$ with respect to an inertial frame, around a given axis. In a reference frame that co-rotates with the fluid, the latter is at rest, and thus governed by the laws of hydrostatics—except that you now have to consider an additional term...

i. Relying on your knowledge from point mechanics, show that the usual equation of hydrostatics (in an inertial frame) is replaced in the co-rotating frame by

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\big[\Phi(\vec{r}) + \Phi_{\text{cen.}}(\vec{r})\big], \tag{1}$$

where $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big[\vec{\Omega}_0 \times \vec{r}\big]^2$ denotes the potential energy from which derives the centrifugal inertial force density, $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$, while $\Phi(\vec{r})$ is the gravitational potential energy.

ii. Show that Eq. (1) implies that the equipotential lines of $\Phi + \Phi_{\text{cen.}}$ coincide with the contours of constant mass density as well as with the isobars.

iii. Consider a slowly spinning fluid planet of mass M, assuming for the sake of simplicity that the mass is concentrated at the planet center, so that the gravitational potential is unaffected by the rotation. Let R_e resp. R_p denote the equatorial resp. polar radius of the planet, where $|R_e - R_p| \ll R_e \simeq R_p$, and g be the gravitational acceleration at the surface of the planet.
Using questions **i.** and **ii.**, show that the difference between the equatorial and polar radii is

$$R_e - R_p \simeq \frac{R_e^2\,|\vec{\Omega}_0|^2}{2g}.$$

Compute this difference in the case of Earth ($R_e \simeq 6.4 \times 10^3$ km)—which as everyone knows behaves as a fluid if you look at it long enough—and compare with the actual value.

13. Stationary vortex:

Let $\vec{\omega}(t,\vec{r}) = A\,\delta(x^1)\,\delta(x^2)\,\vec{e}_3$ be the vorticity field in a fluid, with A a real constant and $\{x^i\}$ Cartesian coordinates. Determine the corresponding flow velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

Hint: You should invoke symmetry arguments and Stokes' theorem. A useful formal analogy is provided by the Maxwell–Ampère equation of magnetostatics.

14. Model of a tornado

In a simplified approach, one may model a tornado as the steady incompressible flow of a perfect fluid—air—with mass density $\rho = 1.3\ \text{kg}\cdot\text{m}^{-3}$, with a vorticity $\vec{\omega}(\vec{r}) = \omega(\vec{r})\,\vec{e}_3$ which remains uniform inside a cylinder—the "eye" of the tornado—with (vertical) axis along $\vec{e}_3$ and a finite radius $a = 50$ m, and vanishes outside.

i. Express the velocity $\mathsf{v}(r) \equiv \big|\vec{\mathsf{v}}(\vec{r})\big|$ at a distance $r = |\vec{r}|$ from the axis as a function of r and and the velocity $\mathsf{v}_a \equiv \mathsf{v}(r\!=\!a)$ at the edge of the eye.
Compute ω inside the eye, assuming $\mathsf{v}_a = 180$ km/h.

ii. Show that for $r > a$ the tornado is equivalent to a vortex at $x^1 = x^2 = 0$ (as in exercise **13**). What is the circulation around a closed curve circling this equivalent vortex?

iii. Assuming that the pressure $\mathcal{P}$ far from the tornado equals the "normal" atmospheric pressure $\mathcal{P}_0$, determine $\mathcal{P}(r)$ for $r > a$. Compute the barometric depression $\Delta\mathcal{P} \equiv \mathcal{P}_0 - \mathcal{P}$ at the edge of the eye. Consider a horizontal roof made of a material with mass surface density 100 kg/m^2: is it endangered by the tornado?

Tutorial sheet 6

Discussion topic: What is a potential flow? What are the corresponding equations of motion?

15. Two-dimensional potential flow. Teapot effect

Consider a steady two-dimensional potential flow with velocity $\vec{\mathsf{v}}(x, y)$, with (x, y) Cartesian coordinates. The associated complex velocity potential is denoted $\phi(z)$, where $z = x + \mathrm{i}y$.

i. Consider the complex potential $\phi(z) = Az^n$ with $A \in \mathbb{R}$ and $n \geq 1/2$. Show that this potential allows you to describe the flow velocity in the sector $\widehat{\mathcal{E}}$ delimited by two walls making an angle $\alpha = \pi/n$.

ii. What can you say about the flow velocity in the vicinity of the end-corner of the sector $\widehat{\mathcal{E}}$?

Hint: Distinguish the cases $\alpha < \pi$ and $\alpha > \pi$.

iii. Teapot effect

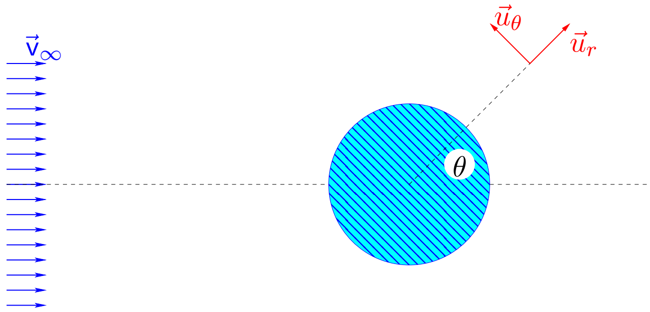
If one tries to pour tea "carefully" from a teapot, one will observe that the liquid will trickle along the lower side of the nozzle, instead of falling down into the cup waiting below. Explain this phenomenon using the flow profile introduced above (in the case $\alpha > \pi$) and the Bernoulli equation.

Literature: Jearl Walker, Scientific American, Oct. 1984 (= Spektrum der Wissenschaft, Feb. 1985).

iv. Assuming now that you are using the potential $\phi(z) = Az^n$ to model the flow of a river, which qualitative behavior can you anticipate for its bank?

16. Potential flow with a vortex. Magnus effect

The purpose of this exercise is to introduce a simplified model for the Magnus effect, which was discussed in the lectures.



One can show that the flow velocity of an incompressible perfect fluid around a cylinder of radius R at rest, with the uniform condition $\vec{\mathsf{v}}(\vec{r}) = \vec{\mathsf{v}}_\infty$ far from the cylinder—$\vec{\mathsf{v}}_\infty$ being perpendicular to the cylinder axis—, is given by

$$\vec{\mathsf{v}}(r, \theta) = \mathsf{v}_\infty \left[\left(1 - \frac{R^2}{r^2}\right) \cos\theta\, \vec{u}_r - \left(1 + \frac{R^2}{r^2}\right) \sin\theta\, \vec{u}_\theta \right], \tag{1}$$

where (r, θ) are polar coordinates—the third dimension (z), along the cylinder axis, plays no role—with the origin at the center of the cylinder (see Figure) and $\vec{u}_r$, $\vec{u}_\theta$ unit length vectors.

One superposes to the velocity field (1) a vortex with circulation Γ, corresponding to a flow velocity

$$\vec{\mathsf{v}}(r, \theta) = \frac{\Gamma}{2\pi r} \vec{u}_\theta. \tag{2}$$

i. Let $C \equiv \Gamma/(4\pi R \mathsf{v}_\infty)$. Determine the points with vanishing velocity for the flow resulting from superposing (1) and (2).

Hint: Distinguish the two cases $C < 1$ and $C > 1$.

ii. How do the streamlines look like in each case? Comment on the physical meaning of the result.

iii. Express the force per unit length $\mathrm{d}\vec{F}/\mathrm{d}z$ exerted on the cylinder by the flow (1)+(2) as function of Γ, v_∞ and the mass density ρ of the fluid.

17. Flow of a liquid in the vicinity of a gas bubble

We assume that the flow of the liquid is radial: $\vec{\mathsf{v}} = \mathsf{v}(t,r)\,\vec{\mathrm{e}}_r$, where the gas bubble is assumed to sit at $\vec{r} = \vec{0}$. Throughout the exercise, the effect of the liquid-gas surface tension—which gives rise to a difference in pressure between both sides of the liquid-gas interface—is neglected.

i. a) Show that the liquid's flow is irrotational. (*Hint*: one can avoid the computation of the curl!)

b) Assuming in addition that the flow is incompressible, derive the expression of $\mathsf{v}(t,r)$ in terms of the bubble radius $R(t)$ and its derivative $\dot{R}(t)$. Deduce therefrom the velocity potential.

ii. One assumes that the gas inside the bubble is an ideal gas which evolves adiabatically when the bubble radius varies, i.e. that its pressure—assumed to be uniform—and volume obey $\mathcal{P}\mathcal{V}^\gamma$ = constant, where γ is the heat capacity ratio. Let $\mathcal{P}_0$ be the value of the pressure at infinity and R_0 the bubble radius when the gas pressure equals $\mathcal{P}_0$.

a) Neglecting the gas flow, give the expression of the pressure inside the bubble in terms of the radius.

b) Writing the Euler equation in terms of the velocity potential, show that $R(t)$ obeys the evolution equation

$$\ddot{R}(t)\,R(t) + \frac{3\big[\dot{R}(t)\big]^2}{2} = \frac{\mathcal{P}_0}{\rho}\left[\left(\frac{R_0}{R(t)}\right)^{3\gamma} - 1\right], \tag{3}$$

where ρ is the liquid mass density.

iii. Suppose now that the bubble radius slightly oscillates about the equilibrium value R_0. Writing $R(t) = R_0[1+\epsilon(t)]$ with $|\epsilon(t)| \ll 1$, derive the (linear!) evolution equation for $\epsilon(t)$. What is the frequency f of such small oscillations?

Numerical application: calculate f for air ($\gamma = 1.4$) bubbles with $R_0 = 1$ mm and $R_0 = 5$ mm in water ($\rho = 10^3$ kg/m^3) for $\mathcal{P}_0 = 10^5$ Pa.

Tutorial sheet 7

Discussion topic: What is a sound wave? How do you derive the corresponding equation of motion? How is the speed of sound defined? What happens when the wave amplitude becomes large?

18. One-dimensional "similarity flow"

Consider a perfect fluid at rest in the region $x \geq 0$ with pressure $\mathcal{P}_0$ and mass density ρ_0; the region $x < 0$ is empty ($\mathcal{P} = 0$, $\rho = 0$). At time $t = 0$, the wall separating both regions is removed, so that the fluid starts flowing into the region $x < 0$. The goal of this exercise is to solve this instance of *Riemann's problem* by determining the flow velocity $\mathsf{v}(t,x)$ for $t > 0$. It will be assumed that the pressure and mass density of the fluid remain related by

$$\frac{\mathcal{P}}{\mathcal{P}_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \qquad \text{with } \gamma > 1$$

throughout the motion. This relation also gives you the speed of sound $c_s(\rho)$.

i. Assume that the dependence on t and x of the various fields involves only the combination $u \equiv x/t$.[1] Show that the continuity and Euler equations can be recast as

$$\big[u - \mathsf{v}(u)\big]\,\rho'(u) = \rho(u)\,\mathsf{v}'(u)$$
$$\rho(u)\big[u - \mathsf{v}(u)\big]\,\mathsf{v}'(u) = c_s^2(\rho(u))\,\rho'(u),$$

where ρ' resp. v' denote the derivative of ρ resp. v with respect to u.

ii. Show that the velocity is either constant, or obeys the equation $u - \mathsf{v}(u) = c_s(\rho(u))$, in which case the squared speed of sound takes the form $c_s^2(\rho) = c_s^2(\rho_0)(\rho/\rho_0)^{\gamma-1}$.

iii. Show that the results of **i.** and **ii.** lead to the relation

$$\mathsf{v}(u) = a + \frac{2}{\gamma - 1}\,c_s(\rho(u)),$$

where a denotes a constant whose value is fixed by the condition that $\mathsf{v}(u)$ remain continuous inside the fluid. Show eventually that in some interval for the values of u, the norm of v is given by

$$|\mathsf{v}(u)| = \frac{2}{\gamma + 1}\big[c_s(\rho_0) - u\big].$$

iv. Sketch the profiles of the mass density $\rho(u)$ and the streamlines $x(t)$ and show that after the removal of the separation at $x = 0$ the information propagates with velocity $2c_s(\rho_0)/(\gamma - 1)$ towards the negative-x region, while it moves to the right with the speed of sound $c_s(\rho)$.

19. Inviscid Burgers equation

The purpose of this exercise is to show how an innocent-looking—yet non-linear—partial differential equation with a smooth initial condition may lead after finite amount of time to a discontinuity, i.e. a shock wave.

Neglecting the pressure term in the one-dimensional Euler equation leads to the so-called *inviscid Burgers equation*

$$\frac{\partial \mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x} = 0.$$

i. Show that the solution with (arbitrary) given initial condition $\mathsf{v}(0,x)$ for $x \in \mathbb{R}$ obeys the implicit equation $\mathsf{v}(0,x) = \mathsf{v}\big(t, x + \mathsf{v}(0,x)\,t\big)$.

Hint: `http://en.wikipedia.org/wiki/Burgers'_equation`

[1] ... which is what is meant by "self-similar".

ii. Consider the initial condition $\mathsf{v}(0,x) = \mathsf{v}_0\,\mathrm{e}^{-(x/x_0)^2}$ with v_0 and x_0 two real numbers. Show that the flow velocity becomes discontinuous at time $t = \sqrt{\mathrm{e}/2}\,x_0/\mathsf{v}_0$, namely at $x = x_0\sqrt{2}$.

20. Heat diffusion

In a dissipative fluid at rest, the energy balance equation becomes

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla}\cdot\left[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\right]$$

with e the internal energy density, κ the heat capacity and T the temperature.

Assuming that $C \equiv \partial e/\partial T$ and κ are constant coefficients and introducing $\chi \equiv \kappa/C$, determine the temperature profile $T(t,\vec{r})$ for $z < 0$ with the boundary condition of a uniform, time-dependent temperature $T(t,z=0) = T_0\cos(\omega t)$ in the plane $z = 0$. At which depth is the amplitude of the temperature oscillations $10\,\%$ of that in the plane $z = 0$?

Tutorial sheet 8

Discussion topic: What are the fundamental equations governing the dynamics of non-relativistic Newtonian fluids?

21. Flow due to an oscillating plane boundary

Consider a rigid infinitely extended plane boundary ($y = 0$) that oscillates in its own plane with a sinusoidal velocity $U \cos(\omega t)\,\vec{e}_x$. The region $y > 0$ is filled with an incompressible Newtonian fluid with uniform kinematic shear viscosity ν. We shall assume that volume forces on the fluid are negligible, that the pressure is uniform and remains constant in time, and that the fluid motion induced by the plane oscillations does not depend on the coordinates x, z.

i. Determine the flow velocity $\vec{\mathsf{v}}(t, y)$ and plot the resulting profile.

ii. What is the characteristic thickness of the fluid layer in the vicinity of the plane boundary that follows the oscillations? Comment on your result.

22. Flow of a Newtonian fluid down a constant slope

A layer of Newtonian fluid is flowing under the influence of gravity (acceleration g) down a slope inclined at an angle α from the horizontal. The fluid itself is assumed to have a constant thickness h, so that its free surface is a plane parallel to its bottom, and the flow is steady, laminar and incompressible. One further assumes that the pressure at the free surface of the fluid as well as "at the ends" at large $|x|$ is constant—i.e., the flow is entirely caused by gravity, not by a pressure gradient.
To fix notations, let x denote the direction along which the fluid flows, with the basis vector oriented downstream, and y be the direction perpendicular to x, oriented upwards.

i. Show that the flow velocity magnitude v and pressure $\mathcal{P}$ of the fluid obey the equations

$$\begin{cases} \dfrac{\partial \mathsf{v}}{\partial x} = 0 \\ \eta \triangle \mathsf{v} = -\rho g \sin\alpha \\ \dfrac{\partial \mathcal{P}}{\partial y} = -\rho g \cos\alpha, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} \mathsf{v} = 0 & \text{at } y = 0 \\ \dfrac{\partial \mathsf{v}}{\partial y} = 0 & \text{at } y = h \\ \mathcal{P} = \mathcal{P}_0 & \text{at } y = h. \end{cases} \tag{2}$$

Determine the pressure and then the velocity profile.

ii. Compute the rate of volume flow ("volumetric flux") across a surface $\mathcal{S}$ perpendicular to the x-direction.

23. Taylor–Couette flow. Measurement of shear viscosity

A Couette viscometer consists of an annular gap, filled with fluid, between two concentric cylinders with height L. The outer cylinder (radius R_2) rotates around the common axis with angular velocity Ω_2, while the inner cylinder (radius R_1) remains motionless. The motion of the fluid is assumed to be two-dimensional, laminar, incompressible, and steady.

Throughout this exercise, we use a system of cylinder coordinates (r, φ, z) with the physicists' usual convention, i.e. the corresponding basis vectors are are normalized to unity.

i. Check that the continuity equation leads to $\mathsf{v}^r = 0$, with v^r the radial component of the flow velocity.

ii. Prove that the Navier–Stokes equation lead to the equations

$$\frac{\mathsf{v}^\varphi(r)^2}{r} = \frac{1}{\rho}\frac{\partial \mathcal{P}(r)}{\partial r} \tag{3}$$

$$\frac{\partial^2 \mathsf{v}^\varphi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \mathsf{v}^\varphi(r)}{\partial r} - \frac{\mathsf{v}^\varphi(r)}{r^2} = 0. \tag{4}$$

What is the meaning of Eq. (3)? Solve Eq. (4) with the ansatz $\mathsf{v}^\varphi(r) = ar + \frac{b}{r}$.

iii. One can show (can you?) that the $r\varphi$-component of the stress tensor is given by

$$\sigma^{r\varphi} = \eta\left(\frac{1}{r}\frac{\partial \mathsf{v}^r}{\partial \varphi} + \frac{\partial \mathsf{v}^\varphi}{\partial r} - \frac{\mathsf{v}^\varphi}{r}\right).$$

Show that $\sigma^{r\varphi} = -\frac{2b\eta}{r^2}$, where b is the same coefficient as above.

iv. A torque $\mathcal{M}_z$ is measured at the surface of the inner cylinder. How can the shear viscosity η of the fluid be deduced from this measurement?
Numerical example: $R_1 = 10$ cm, $R_2 = 11$ cm, $L = 10$ cm, $\Omega_2 = 10$ rad·s^{-1} and $\mathcal{M}_z = 7,246\cdot 10^{-3}$ N·m.

24. Dimensional consideration for viscous flows in a tube

Consider the motion of a given fluid in a cylindrical tube of length L and of circular cross section under the action of a difference $\Delta\mathcal{P}$ between the pressures at the two ends of the tube. The relation between the pressure drop per unit length $\Delta\mathcal{P}/L$ and the magnitude of the mean velocity $\langle \mathsf{v}\rangle$—defined as the average over a cross section of the tube—is given by

$$\frac{\Delta\mathcal{P}}{L} = C\langle \mathsf{v}\rangle^n,$$

with C a constant that depends on the fluid mass density ρ, on the kinematic shear viscosity ν, and on the radius a of the tube cross section. n is a number which depends on the type of flow: $n = 1$ if the flow is laminar (this is the Hagen–Poiseuille law seen in the lecture), while measurements in turbulent flows by Hagen (1854) resp. Reynolds (1883) have given $n = 1.75$ resp. $n = 1.722$.

Assuming that C is—up to a pure number—a product of powers of ρ, ν and a, determine the exponents of these power laws using dimensional arguments.

Tutorial sheet 9

Discussion topic: Dynamical similarity and the Reynolds number. You could also educate yourself on the topic of Life at low Reynolds number and the "scallop theorem" by reading E. M. Purcell's article (also accessible via the web page of the lectures)

25. Equations of fluid dynamics in a uniformly rotating reference frame

This exercise is inspired by Chapter 14.5.1 of *Modern Classical Physics* by Roger D. Blandford and Kip S. Thorne.

For the study of various physical problems (see examples in question **iv.a**), it may be more convenient to study the dynamics of a fluid from a reference frame $\mathcal{R}_{\Omega_0}$ in uniform rotation with angular velocity $\vec{\Omega}_0$ with respect to an inertial frame $\mathcal{R}_0$.

In exercise **12**, you already investigated hydrostatics in a rotating reference frame: in that case only the centrifugal acceleration plays a role, which can be entirely recast as the effect of a potential energy $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big(\vec{\Omega}_0 \times \vec{r}\big)^2$ leading to the centrifugal inertial force density $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$. The purpose of this exercise is to generalize that result to the derivation of (some of) the equations governing a flowing Newtonian fluid.

i. Kinematics

Recall the expressions of the centrifugal and Coriolis accelerations acting on a small fluid element in terms of its position vector $\vec{r}$ and velocity $\vec{\mathsf{v}}$ (measured in $\mathcal{R}_{\Omega_0}$) and of the angular velocity.

ii. Incompressibility condition

Writing down the relation between the velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}_{\Omega_0}$ and that measured in $\mathcal{R}_0$, show that the incompressibility condition valid in the inertial frame leads to $\vec{\nabla}\cdot\vec{\mathsf{v}} = 0$.

iii. Navier–Stokes equation

Show that the incompressible Navier–Stokes equation from the point of view of an observer at rest in the rotating reference frame $\mathcal{R}_{\Omega_0}$ reads (the variables are omitted)

$$\frac{\mathrm{D}\vec{\mathsf{v}}}{\mathrm{D}t} = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}_{\text{eff.}} + \nu\triangle\vec{\mathsf{v}} - 2\vec{\Omega}_0\times\vec{\mathsf{v}} \qquad (1)$$

where $\mathcal{P}_{\text{eff.}} = \mathcal{P} + \rho\big(\Phi + \Phi_{\text{cen.}}\big)$, with Φ the potential energy from which (non-inertial) volume forces acting on the fluid derive. Check that you recover the equation of hydrostatics found in exercise **12**.

iv. Dimensionless numbers and limiting cases

a) Let L_c resp. v_c denote a characteristic length resp. velocity for a given flow. The Ekman and Rossby numbers are respectively defined as

$$\text{Ek} \equiv \frac{\nu}{|\Omega_0|L_c^2} \quad , \quad \text{Ro} \equiv \frac{\mathsf{v}_c}{|\Omega_0|L_c}.$$

Compute Ek and Ro in a few numerical examples:
– $L_c \approx 100$ km, $\mathsf{v}_c \approx 10\ \text{m}\cdot\text{s}^{-1}$, $\Omega_0 \approx 10^{-4}\ \text{rad}\cdot\text{s}^{-1}$, $\nu \approx 10^{-5}\ \text{m}^2\cdot\text{s}^{-1}$ (wind in the Earth atmosphere);
– $L_c \approx 1000$ km, $\mathsf{v}_c \approx 0.1\ \text{m}\cdot\text{s}^{-1}$, $\Omega_0 \approx 10^{-4}\ \text{rad}\cdot\text{s}^{-1}$, $\nu \approx 10^{-6}\ \text{m}^2\cdot\text{s}^{-1}$ (ocean stream);
– $L_c \approx 10$ cm, $\mathsf{v}_c \approx 1\ \text{m}\cdot\text{s}^{-1}$, $\Omega_0 \approx 10\ \text{rad}\cdot\text{s}^{-1}$, $\nu \approx 10^{-6}\ \text{m}^2\cdot\text{s}^{-1}$ (coffee/tea in your cup).

b) Assuming stationarity, which term in Eq. (1) is negligible (against which) at small Ekman number? at small Rossby number?

Write down the simplified equation of motion valid when both Ek $\ll 1$ and Ro $\ll 1$ (to which of the above examples does this correspond?). How do the (effective) pressure gradient $\vec{\nabla}\mathcal{P}_{\text{eff.}}$ and flow velocity stand relative to each other?

26. Vortex dynamics in Newtonian fluids

i. Show that in a barotropic fluid with only conservative forces, the vorticity $\vec{\omega}$ is governed by

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} - \vec{\nabla}\times\big[\vec{\mathrm{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})\big] = \frac{\eta}{\rho(t,\vec{r})}\triangle\vec{\omega}(t,\vec{r}). \tag{2}$$

ii. Diffusion of a rectilinear vortex

Consider the incompressible flow (with constant uniform ρ) with at $t=0$ a rectilinear vortex

$$\vec{\omega}(t{=}0,\vec{r}) = \frac{\Gamma_0}{2\pi r}\delta(z)\,\vec{\mathrm{e}}_z \tag{3}$$

along the z-axis. The system geometry suggests the use of cylindrical coordinates (r,θ,z).

a) Assuming (why does this make sense?) that at $t>0$ the vorticity is still along the z-direction and only depends on the distance r from the axis: $\vec{\omega}(t,\vec{r}) = \omega^z(t,r)\vec{\mathrm{e}}_z$, show that Eq. (2) simplifies to a (known) partial differential equation for ω^z.

b) Can you solve this differential equation with the initial condition (3)?[1] You should find that at time t the vorticity extends over a region of typical width $\sqrt{4\eta t/\rho}$.

c) Assuming you obtained $\omega^z(t,r)$ at the previous step, you can now compute the circulation of the velocity field around a circle of radius R centered on the z-axis. You should find

$$\Gamma(t,R) = \Gamma_0\big[1-\mathrm{e}^{-\rho R^2/(4\eta t)}\big]. \tag{4}$$

Comment on this result (*Hint*: compare with the lecture of May 4th).

[1] One possibility is to remember the lecture of May 25th, in particular the discussion of heat diffusion.

Tutorial sheet 10

27. Flows at small Reynolds number

Check that an alternative form of the Stokes equation for creeping incompressible flows is[1]

$$\sum_{j=1}^{3} \frac{\partial \sigma^{ij}}{\partial x^j} = 0 \quad \forall i \in \{1,2,3\} \tag{1}$$

where σ^{ij} denotes the components (in a system of Cartesian coordinates) of the stress tensor — whose expression you can find in § III.3.3.b of the lecture notes.

28. Instability of the viscous Burgers equation

Neglecting the pressure term in the Navier–Stokes equation for a one-dimensional incompressible problem without external force yields the so-called *viscous Burgers equation*[2]

$$\frac{\partial \mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x} = \nu \frac{\partial^2 \mathsf{v}(t,x)}{\partial x^2}, \tag{2}$$

where $\nu \equiv \eta/\rho$ is the kinematic shear viscosity of the fluid. A trivial solution to this equation of motion is the steady uniform flow $\mathsf{v}(t,x) = \mathsf{v}_0$.

Let us add a perturbation $\delta\mathsf{v}(t,x)$.

a) Write down the linearized equation of motion governing the evolution of $\delta\mathsf{v}$ and derive the corresponding dispersion relation using an appropriate Fourier ansatz.

b) Fixing first $k \in \mathbb{R}$, check that the perturbation is exponentially damped in time.

c) Consider now a fixed $\omega \in \mathbb{R}$. How does the perturbation along the x-direction? (*Hint*: For the sake of simplicity you may restrict your discussion to the small-viscosity case $\omega\nu \ll \mathsf{v}_0^2$.)

29. Instabilities in parallel shear flows

In the lectures we considered a number of simple steady incompressible flows with velocity of the form $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(y)\vec{e}_x$, where x, y, z are Cartesian coordinates. For the stability of such so-called "parallel shear flows" there exist a number of results, some of which are discussed in this exercise. Throughout we assume that the mass density ρ_0 remains uniform and constant, and that there are no external forces.

i. Starting from the continuity and incompressible Navier–Stokes equations, write down the linearized equations of motion governing the evolution of perturbations $\delta\vec{\mathsf{v}}(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ of steady fields $\vec{\mathsf{v}}_0(\vec{r})$ and $\mathcal{P}_0(\vec{r})$, assuming $\vec{\mathsf{v}}_0(\vec{r}) = \mathsf{v}_0(y)\vec{e}_x$.

One can show (Squire's theorem) that it is sufficient to investigate perturbations that are two-dimensional, i.e. that do not depend on z and such that $\delta\vec{\mathsf{v}}$ lies in the (x,y)-plane. To describe the latter, one can introduce the associated stream function $\psi(t,\vec{r})$, such that the non-zero components of $\delta\vec{\mathsf{v}}$ are given by $\delta\mathsf{v}^x = -\partial\psi/\partial y$ and $\delta\mathsf{v}^x = \partial\psi/\partial x$.

ii. Assume first that the fluid is perfect.

a) Using the linearized equations of motion you obtained in **i.**, show that the stream function satisfies the partial differential equation

$$\left[\frac{\partial}{\partial t} + \mathsf{v}_0(y)\frac{\partial}{\partial x}\right]\triangle\psi(t,\vec{r}) - \frac{\partial^2 \mathsf{v}_0(y)}{\partial y^2}\frac{\partial \psi(t,\vec{r})}{\partial x} = 0. \tag{3}$$

[1] A shorter (and thus more elegant?) form is $\vec{\nabla}\cdot\boldsymbol{\sigma} = \vec{0}$.

[2] You already encountered its inviscid version in exercise **19.**

b) Making the Fourier ansatz $\psi(t,\vec{r}) = \widetilde{\psi}(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)}$, show that Eq. (2) leads to *Rayleigh's equation*

$$\big[\mathsf{v}_0(y) - c(k)\big]\bigg(\frac{\partial^2}{\partial y^2} - k^2\bigg)\widetilde{\psi}(y) - \frac{\partial^2 \mathsf{v}_0(y)}{\partial y^2}\widetilde{\psi}(y) = 0, \tag{4}$$

where $c(k) \equiv \omega/k$.

For a given profile $\mathsf{v}_0(y)$ of the unperturbed flow and a fixed wavenumber k, this is an eigenvalue equation, whose solutions are eigenfunctions $\widetilde{\psi}(y)$ with associated eigenvalues $c(k)$. Show that if $\widetilde{\psi}$ is an eigenfunction associated with some eigenvalue $c(k)$, then its complex conjugate $\widetilde{\psi}^*$ is also an eigenfunction, with eigenvalue $c(k)^*$. What does this mean for the stability of the unperturbed flow in case one of the eigenvalues is not real?

iii. If you still have time, you may show that in a Newtonian incompressible fluid, Rayleigh's equation is replaced by the *Orr–Sommerfeld equation*

$$\big[\mathsf{v}_0(y) - c(k)\big]\bigg(\frac{\partial^2}{\partial y^2} - k^2\bigg)\widetilde{\psi}(y) - \frac{\partial^2 \mathsf{v}_0(y)}{\partial y^2}\widetilde{\psi}(y) = \frac{\nu}{\mathrm{i}k}\bigg(\frac{\partial^2}{\partial y^2} - k^2\bigg)^2\widetilde{\psi}(y), \tag{5}$$

with $\nu \equiv \eta/\rho_0$ the kinematic shear viscosity of the fluid.

Tutorial sheet 11

Discussion topic: Turbulence in fluids: what is it? Why does it require a Reynolds number larger than some critical value to develop? In fully developed turbulence, what are the mean flow, the fluctuating flow, the Reynolds stress tensor, the energy cascade?

30. A mathematical model to reproduce some features of fully developed turbulence

While trying to solve the problem of incompressible turbulence in fluids, Burgers wrote down a system of simpler equations—a toy mathematical model—that share a few features of the dynamical equations governing the mean flow and the flow fluctuations, namely

$$\frac{\mathrm{d}\bar{\mathsf{v}}(t)}{\mathrm{d}t} = \mathcal{P} - \mathsf{v}'(t)^2 - \nu\,\bar{\mathsf{v}}(t), \tag{1a}$$

$$\frac{\mathrm{d}\mathsf{v}'(t)}{\mathrm{d}t} = \bar{\mathsf{v}}(t)\mathsf{v}'(t) - \nu\,\mathsf{v}'(t), \tag{1b}$$

with $\bar{\mathsf{v}}$, v' two unknown functions, while ν is a parameter and $\mathcal{P}$ a constant. In these equations, all quantities (including t) are dimensionless and real.
The questions **i.**, **ii.**, **iii.**, **iv.** are to a very large extent independent from each other.

i. Enumerate the similarities between Burgers' set of equations and the "true" ones given in the lecture. That is, identify the physical content of each term in Eqs. (1), and recognize how key mathematical features of the fluid dynamical equations are seemingly reproduced—while others are obviously not, which may deserve a discussion as well.

ii. Viewing $\bar{\mathsf{v}}$, v' as velocities, write down the differential equation governing the evolution of the sum of the associated kinetic energies (per unit mass...). Note that the terms which you obtain have a straightforward physical interpretation, which smoothly matches those found in question **i.**

iii. "Laminar" solution

a) Show that equations (1) admit a set of stationary solutions with a finite "mean flow velocity" $\bar{\mathsf{v}} = \bar{\mathsf{v}}_0$ and a vanishing "fluctuating velocity" v'.

b) Check that these solutions are stable as long as $\mathcal{P} < \nu^2$. That is, any perturbation $(\delta\bar{\mathsf{v}}, \delta\mathsf{v}')$ yielding total velocities $\bar{\mathsf{v}}(t) = \bar{\mathsf{v}}_0 + \delta\bar{\mathsf{v}}(t)$, $\mathsf{v}'(t) = \delta\mathsf{v}'(t)$ will be exponentially damped. On the other hand, the solution $(\bar{\mathsf{v}} = \bar{\mathsf{v}}_0, \mathsf{v}' = 0)$ is unstable for $\mathcal{P} > \nu^2$.

iv. "Turbulent" solution

Let us now assume $\mathcal{P} > \nu^2$.

a) Show that equations (1) now admit two sets of stationary solutions, both involving a finite mean flow velocity $\bar{\mathsf{v}}$—the same for both sets—and a finite fluctuating velocity $\mathsf{v}' = \pm\mathsf{v}'_0$.

b) Show that both solutions are stable for $\mathcal{P} > \nu^2$.

Hint: You should have to distinguish two cases, namely $\nu < \mathcal{P} \leq \frac{9}{8}\nu^2$ and $\mathcal{P} > \frac{9}{8}\nu^2$.

The appearance of several regimes—one laminar ($\mathsf{v}' = 0$), the other turbulent ($\mathsf{v}' \neq 0$)—depending on the value of a parameter is reminiscent of the onset of turbulence above a geometry-dependent given Reynolds number in the real fluid dynamical case: in that respect, Burgers' toy model reproduces an important feature of the true equations. On the other hand, the existence of two competing turbulent solutions above the critical parameter value is an over-simplification of the real turbulent motion.

31. Dynamics of the mean flow in fully developed turbulence

The velocity field resp. pressure for an incompressible turbulent flow is split into an average and a fluctuating part as

$$\vec{\mathsf{v}}(t,\vec{r}) = \overline{\vec{\mathsf{v}}}(t,\vec{r}) + \vec{\mathsf{v}}'(t,\vec{r}) \quad \text{resp.} \quad \mathcal{P}(t,\vec{r}) = \overline{\mathcal{P}}(t,\vec{r}) + \mathcal{P}'(t,\vec{r}),$$

where the motion with $\overline{\vec{\mathsf{v}}}$, $\overline{\mathcal{P}}$ is referred to as "mean flow". For the sake of simplicity, a system of Cartesian coordinates is being assumed—the components of the gradient thus involve partial derivatives, instead of the more general covariant derivatives. Throughout the exercise, Einstein's summation convention over repeated indices is used.

Check that the incompressible Navier–Stokes equation obeyed by $\vec{\mathsf{v}}$ and $\mathcal{P}$ leads for the mean-flow quantities to the equation

$$\frac{\partial\overline{\mathsf{v}^i}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{\mathsf{v}^i} = -\frac{1}{\rho}\frac{\partial\overline{\mathcal{P}}}{\partial x_i} - \frac{\partial\overline{\mathsf{v}'^i\mathsf{v}'^j}}{\partial x^j} + \nu\triangle\overline{\mathsf{v}^i}. \tag{2}$$

Show that this gives for the kinetic energy per unit mass $\overline{k} \equiv \frac{1}{2}\overline{\vec{\mathsf{v}}}^2$ associated with the mean flow the evolution equation

$$\frac{\partial\overline{k}}{\partial t} + \big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\overline{k} = -\frac{\partial}{\partial x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}}\overline{\mathsf{v}^j} + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathsf{v}_i}\right] + \Big(\overline{\mathsf{v}'^i\mathsf{v}'^j} - 2\nu\overline{\mathbf{S}^{ij}}\Big)\overline{\mathbf{S}_{ij}} \tag{3}$$

with $\overline{\mathbf{S}^{ij}} \equiv \dfrac{1}{2}\left(\dfrac{\partial\overline{\mathsf{v}^i}}{\partial x_j} + \dfrac{\partial\overline{\mathsf{v}^j}}{\partial x_i} - \dfrac{2}{3}g^{ij}\vec{\nabla}\cdot\overline{\vec{\mathsf{v}}}\right)$ the components of the (mean) rate-of-shear tensor.

Tutorial sheet 12

Discussion topic: Convective heat transfer: what is the Rayleigh–Bénard convection? Describe its phenomenology. Which effects play a role?

For the sake of brevity, throughout this exercise sheet the dependence of the various fields on the space and time variables is not written.

32. Thermal convection between two vertical plates

Consider a fluid in a gravitational potential $-\vec{\nabla}\Phi = \vec{g} \equiv g\,\vec{e}_z$, contained between two infinite vertical plates at $x = \pm d/2$. When the plates have the same uniform temperature, there exist a static "isothermal" solution of the equations of motion describing the fluid, in which the latter is at the same temperature $T_{\rm eq}$ everywhere.

Assume that the plate at $x = -d/2$ resp. $x = +d/2$ is at a uniform temperature T_- resp. T_+ with $T_- < T_+$: this will induce a motion of the fluid, which we want to investigate. For simplicity, we shall assume that the motion is steady, and that it constitutes a small perturbation of the equilibrium state in which both temperatures are equal. Accordingly, the pressure, temperature and mass density are written in the form

$$\mathcal{P} = \mathcal{P}_{\rm eq} + \delta\mathcal{P}\ ,\ T = T_{\rm eq} + \delta T\ ,\ \rho = \rho_{\rm eq} + \delta\rho, \tag{1}$$

where the quantities with the subscript eq. refer to the equilibrium state, which need not be further specified.

i. Show that the relevant equations (IX.8), (IX.9), (IX.12), (IX.13) of the lecture notes lead for the small quantities $\delta\mathcal{P}$, δT, $\delta\rho$ and $\vec{\mathsf{v}}$ to the system

$$\vec{\nabla}\cdot\vec{\mathsf{v}} = 0 \quad \text{(2a)} \qquad \vec{\nabla}\big(\delta\mathcal{P}\big) = \delta\rho\,\vec{g} + \nu\rho_{\rm eq}\triangle\vec{\mathsf{v}} \quad \text{(2b)}$$

$$\vec{\mathsf{v}}\cdot\vec{\nabla}T_{\rm eq} = \alpha\triangle\big(\delta T\big) \quad \text{(2c)} \qquad \delta\rho = -\alpha_{(\mathcal{V})}\rho_{\rm eq}\,\delta T \quad \text{(2d)}$$

where the stationarity assumption has already been used. How did you implement the assumed smallness of the "perturbations" to the static state? How can you already simplify Eq. (2c)?

ii. Let us assume that the new flow only depends on the x-coordinate, and that the y-direction plays no role at all; in particular, there is no component v_y. Let us further assume that the net mass flow through any plane $z =$ const. vanishes, i.e.

$$\int_{-d/2}^{d/2} \rho_{\rm eq}\,\mathsf{v}_z(x,y,z)\,{\rm d}x = 0 \tag{3}$$

for all y, z: this condition allows us to fully specify the "boundary" conditions obeyed by the velocity field.

a) Determine first the temperature-variation profile $\delta T(x)$ and deduce from it the mass density perturbation $\delta\rho(x)$. (*Hint*: Eqs. (2c)–(2d)).

b) Determine the velocity profile between the two plates. How do the streamlines look like?

iii. Time for some physics: what is absurd with the assumption of an infinite extent in the z-direction? Is there really heat convection in the flow determined in question **ii.**? Can you think of an (everyday-life) example—with finite plates!—corresponding to the setup considered here?

33. (1+1)-dimensional relativistic motion

On June 29th, the flow velocities considered in the lectures will reach the relativistic regime. To prepare for this event, you may refresh your knowledge on Special Relativity. This exercise is here to help you in that direction, and also introduces coordinates which will be used later in the lectures.

Consider a (1+1)-dimensional relativistic motion along a direction denoted as z, where the denomination "1+1" stands for one time and one spatial dimension. Throughout the exercise, the other two spatial directions play no role and the corresponding variables x, y are totally omitted. In addition, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein's summation convention over repeated indices.

To describe the physics, one may naturally use Minkowski coordinates $(x^0, x^3) = (t, z)$, with corresponding derivatives $(\partial_0, \partial_3) = (\partial/\partial t, \partial/\partial z)$. If there is a high-velocity motion in the z-direction, a better choice might be to use the *proper time* τ and *spatial rapidity* ς such that[1]

$$x^{0'} \equiv \tau \equiv \sqrt{t^2 - z^2}, \quad x^{3'} \equiv \varsigma \equiv \frac{1}{2} \log \frac{t+z}{t-z} \quad \text{where } |z| \leq t. \tag{4}$$

The partial derivatives with respect to these new coordinates will be denoted $(\partial_{0'}, \partial_{3'}) = (\partial/\partial\tau, \partial/\partial\varsigma)$.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau \cosh\varsigma, \quad z = \tau \sinh\varsigma. \tag{5}$$

(*Hint:* Recognize $\frac{1}{2}\log\frac{1+u}{1-u}$).

ii. In a change of coordinates $\{x^\mu\} \to \{x^{\mu'}\}$, the contravariant components V^μ of a 4-vector transform according to $V^\mu \to V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$ (with summation over ν!) where $\Lambda^{\mu'}{}_\nu \equiv \partial x^{\mu'}/\partial x^\nu$.

Compute first from Eq. (5) the matrix elements $\Lambda^\nu{}_{\mu'} \equiv \partial x^\nu/\partial x^{\mu'}$ (with $\nu \in \{0, 3\}$, $\mu' \in \{0', 3'\}$) of the inverse transformation $\{V^{\mu'}\} \to \{V^\mu\}$. Inverting the 2×2-matrix you thus found, deduce the following relationship between the components of the 4-vector in the two coordinate systems

$$\begin{cases} V^{0'} = \cosh\varsigma\, V^0 - \sinh\varsigma\, V^3 \\ V^{3'} = -\dfrac{1}{\tau}\sinh\varsigma\, V^0 + \dfrac{1}{\tau}\cosh\varsigma\, V^3. \end{cases} \tag{6}$$

iii. Using the relation $\partial_\nu = \Lambda^{\mu'}{}_\nu \partial_{\mu'}$ and the matrix elements $\{\Lambda^{\mu'}{}_\nu\}$ you found in **ii.**—and which can be read off Eq. (6)—, express the "4-divergence" $\partial_\nu V^\nu$ of a 4-vector field V^ν in terms of the partial derivatives $\partial_{\mu'}$ and the components $V^{\mu'}$ in the (τ, ς)-system.

You should find a result that does not equal $\partial_{\mu'} V^{\mu'} = \partial_\tau V^\tau + \partial_\varsigma V^\varsigma$, which is why in the lecture notes the notation $\mathrm{d}_{\mu'} V^{\mu'}$ is used for the 4-divergence in an arbitrary coordinate system.

iv. Draw on a spacetime diagram—with t on the vertical axis and z on the horizontal axis—the lines of constant τ and those of constant ς.

Remark: The coordinates (τ, ς) are sometimes called *Milne coordinates*.

[1] ς = `\varsigma` is the word-final form for the lower case sigma, not to be confused with ζ (zeta).

Tutorial sheet 13

Discussion topic: What are the fundamental equations of the dynamics of a relativistic fluid? What is the relation between the energy-momentum tensor of a perfect relativistic fluid and its internal energy, pressure, and four-velocity? How is the latter defined?

Hint: If the covariant derivatives d_μ in the following exercises upset you, choose Minkowski coordinates, in which $\mathrm{d}_\mu = \partial_\mu$.

34. Quantum number conservation

Consider a 4-current with components $N^\mu(x)$ obeying the continuity equation $\mathrm{d}_\mu N^\mu(\mathsf{x}) = 0$. Show that the quantity $\mathcal{N} = \int N^0(\mathsf{x})\,\mathrm{d}^3\vec{r}/c$ is a Lorentz scalar, by convincing yourself first that $\mathcal{N}$ can be rewritten in the form

$$\mathcal{N} = \frac{1}{c}\int_{x^0=\text{const.}} N^\mu(\mathsf{x})\,\mathrm{d}^3\sigma_\mu, \tag{1}$$

where $\mathrm{d}^3\sigma_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ is a 4-vector, with $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ the antisymmetric 4-tensor defined by

$$\mathrm{d}^3\mathcal{V}^{012} = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2, \quad \mathrm{d}^3\mathcal{V}^{021} = -\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^1, \quad \text{etc.}$$

and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric Levi–Civita tensor with the convention $\epsilon_{0123} = +1$, such that $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ represents the 3-dimensional hypersurface element in Minkowski space.

35. Energy-momentum tensor

Let $\mathcal{R}$ denote a fixed reference frame. Consider a perfect fluid whose local rest frame at a point x moves with velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}$. Show with the help of a Lorentz transformation that the Minkowski components of the energy-momentum tensor of the fluid at x are given to order $\mathcal{O}(|\vec{\mathsf{v}}|/c)$ by

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^1}{c} & \mathcal{P} & 0 & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^2}{c} & 0 & \mathcal{P} & 0 \\ (\epsilon+\mathcal{P})\frac{\mathsf{v}^3}{c} & 0 & 0 & \mathcal{P} \end{pmatrix},$$

where for the sake of brevity the x-dependence of the various fields is omitted. Check the compatibility of this result with the general formula for $T^{\mu\nu}$ given in the lecture.

36. Equations of motion of a perfect relativistic fluid

In this exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

i. Check that the tensor with components $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ defines a projector on the subspace orthogonal to the 4-velocity.
Denoting by d_μ the components of the (covariant) 4-gradient, we define $\nabla^\nu \equiv \Delta^{\mu\nu}\mathrm{d}_\mu$. Can you see the rationale behind this notation?

ii. Show that the energy-momentum conservation equation for a perfect fluid is equivalent to the two equations

$$u^\mu \mathrm{d}_\mu \epsilon + (\epsilon+\mathcal{P})\mathrm{d}_\mu u^\mu = 0 \quad \text{and} \quad (\epsilon+\mathcal{P})u^\mu \mathrm{d}_\mu u^\nu + \nabla^\nu \mathcal{P} = 0. \tag{2}$$

Which known equation does the second one evoke?

37. A family of solutions of the dynamical equations for perfect relativistic fluids

Let $\{x^\mu\}$ denote Minkowski coordinates and $\tau^2 \equiv -x^\mu x_\mu$, where the "mostly plus" metric is used. Show that the following four-velocity, pressure and charge density constitute a solution of the equations describing the motion of a perfect relativistic fluid with equation of state $\mathcal{P} = K\varepsilon$ and a single conserved charge:

$$u^\mu(\mathsf{x}) = \frac{x^\mu}{\tau} \quad , \quad \mathcal{P}(\mathsf{x}) = \mathcal{P}_0\left(\frac{\tau_0}{\tau}\right)^{3(1+K)} \quad , \quad n(\mathsf{x}) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}\big(\sigma(\mathsf{x})\big), \tag{3}$$

with τ_0, $\mathcal{P}_0$, n_0 arbitrary constants and $\mathcal{N}$ an arbitrary function of a single argument, while σ is a function of spacetime coordinates with vanishing comoving derivative: $u^\mu \partial_\mu \sigma(\mathsf{x}) = 0$.