X.4 Dissipative relativistic fluids

In a dissipative relativistic fluid, the transport of conserved charges and 4-momentum is not only convective—i.e. caused by the fluid motion—, but may also be diffusive, due e.g. to spatial gradients of the flow velocity, the temperature, or the chemical potentials associated with the charges. The description of these new types of transport necessitates the introduction of additional contributions to the charge-number 4-currents and the energy-momentum tensor (§ X.4.1), that break the local spatial isotropy of the fluid. As a matter of fact, the local rest frame of the fluid is no longer uniquely determined, but there are in general different choices that lead to "simple" expressions for the dynamical quantities (§ X.4.2).

Adopting one of these particular reference frames as local rest frame, the general dynamical equations describing the motion of a dissipative relativistic fluid are derived in § X.4.3. Eventually, Secs. X.4.4 and X.4.5...

For the sake of brevity, we adopt in this Section a "natural" system of units in which the speed of light c and the Boltzmann constant k_B equal 1.

X.4.1 Dissipative currents

To account for the additional types of transport present in dissipative fluids, extra terms are added to the charge-number 4-currents and the energy-momentum tensor. Denoting with a subscript perf. the quantities for a perfect fluid, their counterparts in the dissipative case thus read

$$\mathsf{N}_a(\mathsf{x}) = \mathsf{N}_{a,\text{perf.}}(\mathsf{x}) + \mathsf{n}_a(\mathsf{x}) \quad , \quad \mathbf{T}(\mathsf{x}) = \mathbf{T}_{\text{perf.}}(\mathsf{x}) + \boldsymbol{\tau}(\mathsf{x}) \tag{X.31a}$$

or equivalently, in terms of the components on a given coordinate system

$$N_a^\mu(\mathsf{x}) = N_{a,\text{perf.}}^\mu(\mathsf{x}) + n_a^\mu(\mathsf{x}) \quad , \quad T^{\mu\nu}(\mathsf{x}) = T_{\text{perf.}}^{\mu\nu}(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x}). \tag{X.31b}$$

In these equations, $\mathsf{N}_a(\mathsf{x})$ resp. $\boldsymbol{\tau}(\mathsf{x})$ is a 4-vector resp. a 4-tensor of degree 2, with components $n_a^\mu(\mathsf{x})$ resp. $\tau^{\mu\nu}(\mathsf{x})$, that represents a dissipative charge-number resp. energy-momentum flux density.

As in the perfect-fluid case, it is natural to introduce a 4-velocity $\mathsf{u}(\mathsf{x})$ in terms of which the quantities $\mathsf{n}_{a,\text{perf.}}(\mathsf{x})$, $\mathbf{T}_{\text{perf.}}(\mathsf{x})$ have a simple, "isotropic" expression. Accordingly, let $\mathsf{u}(\mathsf{x})$ be an at first *arbitrary* time-like 4-vector field with constant magnitude -1,[66] with components $u^\mu(\mathsf{x})$, $\mu \in \{0,1,2,3\}$. The reference frame in which the spatial components of this 4-velocity vanish will constitute the local rest frame LR(x) associated with $\mathsf{u}(\mathsf{x})$.

The projector $\boldsymbol{\Delta}$ on the 3-dimensional space orthogonal to the 4-velocity $\mathsf{u}(\mathsf{x})$ is defined as in Eq. (X.19b), i.e. its components are[66]

$$\Delta^{\mu\nu}(\mathsf{x}) \equiv g^{\mu\nu}(\mathsf{x}) + u^\mu(\mathsf{x})u^\nu(\mathsf{x}), \tag{X.32}$$

with $g^{\mu\nu}(\mathsf{x})$ the components of the inverse metric tensor $\mathbf{g}^{-1}(\mathsf{x})$. For the comprehension it is important to realize that $\boldsymbol{\Delta}$ plays the role of the identity in the 3-space orthogonal to $\mathsf{u}(\mathsf{x})$.

By analogy with Eqs. (X.17a), (X.18), and (X.19a), one thus writes

$$\boxed{N_a^\mu(\mathsf{x}) = \mathcal{n}_a(\mathsf{x})u^\mu(\mathsf{x}) + n_a^\mu(\mathsf{x})} \tag{X.33a}$$

or equivalently

$$\boxed{\mathsf{N}_a(\mathsf{x}) = \mathcal{n}_a(\mathsf{x})\mathsf{u}(\mathsf{x}) + \mathsf{n}_a(\mathsf{x})} \tag{X.33b}$$

and

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x})u^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \mathcal{P}(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}) + \tau^{\mu\nu}(\mathsf{x})} \tag{X.34a}$$

[66] since $c^2 = 1$

i.e., in component-free form

$$\boxed{\mathsf{T}(\mathsf{x}) = \epsilon(\mathsf{x})\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \mathcal{P}(\mathsf{x})\boldsymbol{\Delta}(\mathsf{x}) + \boldsymbol{\tau}(\mathsf{x}).} \tag{X.34b}$$

Remark: It should be kept in mind that the various fields $\epsilon(\mathsf{x})$, $\mathcal{P}(\mathsf{x})$, ... in Eqs. (X.33)–(X.34) depend on the still arbitrary four-velocity $\mathsf{u}(\mathsf{x})$,

The precise physical content and mathematical form of the additional terms can now be further specified.

Tensor algebra

In order for $n_a(\mathsf{x})$ to represent the comoving charge density, the dissipative 4-vector $\mathsf{n}_a(\mathsf{x})$ can have no timelike component in the local rest frame LR(x) defined by the fluid 4-velocity, see definition (X.12). Accordingly, the condition

$$u_\mu(\mathsf{x})n_a^\mu(\mathsf{x})\big|_{\mathrm{LR(x)}} = 0$$

must hold in the local rest frame. Since the term on the left hand side of this identity is a Lorentz scalar, its value remains the same in any reference frame or coordinate system, i.e.

$$u_\mu(\mathsf{x})n_a^\mu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\mathsf{n}_a(\mathsf{x}) = 0. \tag{X.35a}$$

Equations (X.33) thus represent the decomposition of a 4-vector in a component parallel to the flow 4-velocity and a component orthogonal to it. Accordingly, one can write

$$n_a^\mu(\mathsf{x}) = \Delta^\mu{}_\nu(\mathsf{x})N_a^\nu(\mathsf{x}). \tag{X.35b}$$

Physically, the 4-vector $\mathsf{n}_a(\mathsf{x})$ represents a *diffusive charge-number 4-current* in the local rest frame, which describes the non-convective transport of the conserved charge of type a.

Similarly, the dissipative energy-momentum current $\boldsymbol{\tau}(\mathsf{x})$ can have no 00-component in the local rest frame, to ensure that $T^{00}(\mathsf{x})$ in that frame still defines the comoving energy density $\epsilon(\mathsf{x})$. This means that the components $\tau^{\mu\nu}(\mathsf{x})$ may not be proportional to the product $u^\mu(\mathsf{x})u^\nu(\mathsf{x})$. The most general symmetric tensor of degree 2 which satisfies that requirement is of the form

$$\tau^{\mu\nu}(\mathsf{x}) = q^\mu(\mathsf{x})u^\nu(\mathsf{x}) + q^\nu(\mathsf{x})u^\mu(\mathsf{x}) + \pi^{\mu\nu}(\mathsf{x}), \tag{X.36a}$$

with $q^\mu(\mathsf{x})$ resp. $\pi^{\mu\nu}(\mathsf{x})$ the components of a 4-vector $\mathsf{q}(\mathsf{x})$ resp. of a $\binom{2}{0}$-type tensor $\boldsymbol{\pi}(\mathsf{x})$ such that

$$u_\mu(\mathsf{x})q^\mu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\mathsf{q}(\mathsf{x}) = 0 \tag{X.36b}$$

and

$$u_\mu(\mathsf{x})\pi^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\boldsymbol{\pi}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}) = 0. \tag{X.36c}$$

Condition (X.36b) expresses the orthogonality of the 4-velocity $\mathsf{u}(\mathsf{x})$ and the 4-vector $\mathsf{q}(\mathsf{x})$, which physically represents the *heat current* or *energy flux density* in the local rest frame.

In turn, the symmetric tensor $\boldsymbol{\pi}(\mathsf{x})$ can be decomposed into the sum of a traceless symmetric tensor $\boldsymbol{\varpi}(\mathsf{x})$ with components $\varpi^{\mu\nu}(\mathsf{x})$ and a tensor proportional to the projector (X.19b) orthogonal to the 4-velocity

$$\pi^{\mu\nu}(\mathsf{x}) = \varpi^{\mu\nu}(\mathsf{x}) + \Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}). \tag{X.36d}$$

The tensor $\boldsymbol{\varpi}(\mathsf{x})$ is the *shear stress tensor* in the local rest frame of the fluid, that describes the transport of momentum due to shear deformations. Eventually, $\Pi(\mathsf{x})$ represents a *dissipative pressure* term, since it behaves as the thermodynamic pressure $\mathcal{P}(\mathsf{x})$ as shown by Eq. (X.37) below.

All in all, the components of the energy-momentum tensor in a dissipative relativistic fluid may thus be written as

$$\boxed{T^{\mu\nu}(\mathsf{x}) = \epsilon(\mathsf{x})u^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \big[\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})\big]\Delta^{\mu\nu}(\mathsf{x}) + q^\mu(\mathsf{x})u^\nu(\mathsf{x}) + q^\nu(\mathsf{x})u^\mu(\mathsf{x}) + \varpi^{\mu\nu}(\mathsf{x}),} \tag{X.37a}$$

which in geometric formulation reads

$$\boxed{\mathsf{T}(\mathsf{x}) = \epsilon(\mathsf{x})\mathsf{u}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \big[\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})\big]\boldsymbol{\Delta}(\mathsf{x}) + \mathsf{q}(\mathsf{x})\otimes\mathsf{u}(\mathsf{x}) + \mathsf{u}(\mathsf{x})\otimes\mathsf{q}(\mathsf{x}) + \boldsymbol{\varpi}(\mathsf{x}).} \tag{X.37b}$$

One easily checks the identities

$$q^\mu(\mathsf{x}) = \Delta^{\mu\nu}(\mathsf{x})T_{\nu\rho}(\mathsf{x})u^\rho(\mathsf{x}); \tag{X.38a}$$

$$\varpi^{\mu\nu}(\mathsf{x}) = \frac{1}{2}\bigg[\Delta^\mu{}_\rho(\mathsf{x})\Delta^\nu{}_\sigma(\mathsf{x}) + \Delta^\nu{}_\rho(\mathsf{x})\Delta^\mu{}_\sigma(\mathsf{x}) - \frac{2}{3}\Delta^{\mu\nu}(\mathsf{x})\Delta_{\rho\sigma}(\mathsf{x})\bigg]T^{\rho\sigma}(\mathsf{x}); \tag{X.38b}$$

$$\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x}) = \frac{1}{3}\Delta^{\mu\nu}(\mathsf{x})T_{\mu\nu}(\mathsf{x}), \tag{X.38c}$$

which together with Eq. (X.15)

$$\epsilon(\mathsf{x}) = u_\mu(\mathsf{x})T^{\mu\nu}(\mathsf{x})u_\nu(\mathsf{x}) = \mathsf{u}(\mathsf{x})\cdot\mathsf{T}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}) \tag{X.38d}$$

allow one to recover the various fields in which the energy-momentum tensor has been decomposed.

Remarks:

∗ The energy-momentum tensor comprises 10 unknown independent fields, namely the components $T^{\mu\nu}(\mathsf{x})$ with $\nu \geq \mu$. In the decomposition (X.37), written in the local rest frame, $\epsilon(\mathsf{x})$, $\mathcal{P}(\mathsf{x}) + \Pi(\mathsf{x})$, the spatial components $q^i(\mathsf{x})$ and $\varpi^{ij}(\mathsf{x})$ represent 1+1+3+5=10 equivalent independent fields—out of the 6 components $\varpi^{ij}(\mathsf{x})$ with $j \geq i$, one of the diagonal ones is fixed by the condition on the trace. This in particular shows that the decomposition of the left hand side of Eq. (X.38c) into two terms is as yet premature—the splitting actually requires of an equation of state to properly identify $\mathcal{P}(\mathsf{x})$.

Similarly, the 4 unknown components N_a^μ of the 4-current associated with the conserved charge of type a are expressed in terms of $n_a(\mathsf{x})$ and the three spatial components $n_a^i(\mathsf{x})$, i.e. an equivalent number of independent fields.

∗ Let $a^{\mu\nu}$ denote the (contravariant) components of an arbitrary $\binom{2}{0}$-tensor. One encounters in the literature the various notations

$$a^{(\mu\nu)} \equiv \frac{1}{2}\big(a^{\mu\nu} + a^{\nu\mu}\big),$$

which represents the symmetric part of the tensor,

$$a^{[\mu\nu]} \equiv \frac{1}{2}\big(a^{\mu\nu} - a^{\nu\mu}\big)$$

for the antisymmetric part—so that $a^{\mu\nu} = a^{(\mu\nu)} + a^{[\mu\nu]}$—, and

$$a^{\langle\mu\nu\rangle} \equiv \bigg(\Delta_\rho^{(\mu}\Delta^{\nu)}_\sigma - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\rho\sigma}\bigg)a^{\rho\sigma},$$

which is the symmetrized traceless projection on the 3-space orthogonal to the 4-velocity. Using these notations, the dissipative stress tensor (X.36a) reads

$$\tau^{\mu\nu}(\mathsf{x}) = q^{(\mu}(\mathsf{x})u^{\nu)}(\mathsf{x}) + \varpi^{\mu\nu}(\mathsf{x}) + \Pi(\mathsf{x})\Delta^{\mu\nu}(\mathsf{x}),$$

while Eq. (X.38b) becomes $\varpi^{\mu\nu}(\mathsf{x}) = T^{\langle\mu\nu\rangle}(\mathsf{x})$.

X.4.2 Local rest frames

At a given point in a dissipative relativistic fluid, the conserved charge(s) and the energy may flow in different directions. This can happen in particular because particle–antiparticle pairs, which do not contribute to the net charge density, still transport energy. Another, not exclusive, possibility is that different conserved quantum numbers flow in different directions. In any case, one can in general not find a preferred reference frame in which the local properties of the fluid are isotropic.

As a consequence, there is also no unique “natural” choice for the 4-velocity $\mathsf{u}(\mathsf{x})$ of the fluid motion. On the contrary, several definitions of the flow 4-velocity are possible, which imply varying relations for the dissipative currents, although the physics that is being described remains the same.

X.4.2 a Eckart frame

A first natural possibility, proposed by Eckart[(az)] [44], is to take the 4-velocity proportional to the 4-current $\mathsf{N}(\mathsf{x})$ for a given conserved charge, namely

$$u^\mu_{\text{Eckart}}(\mathsf{x}) \equiv \frac{N^\mu(\mathsf{x})}{\sqrt{-N_\nu(\mathsf{x})N^\nu(\mathsf{x})}}. \tag{X.39}$$

Accordingly, the dissipative charge-number flux $\mathsf{n}(\mathsf{x})$ vanishes automatically, so that the expression of charge conservation is simpler with that choice.
The local rest frame associated with the flow 4-velocity (X.39) is then referred to as *Eckart frame*.

A drawback of that definition of the fluid 4-velocity is that the net charge number can possibly vanish in some regions of a given flow, so that $\mathsf{u}_{\text{Eckart}}(\mathsf{x})$ is not defined unambiguously in such domains.

X.4.2 b Landau frame

An alternative natural definition is that of Landau[(ba)] (and Lifshitz[(bb)]), according to whom the fluid 4-velocity is taken to be proportional to the energy flux density. The corresponding 4-velocity is defined by the implicit equation

$$u^\mu_{\text{Landau}}(\mathsf{x}) = \frac{T^\mu{}_\nu(\mathsf{x})\,u^\nu_{\text{Landau}}(\mathsf{x})}{\sqrt{-u^\lambda_{\text{Landau}}(\mathsf{x})\,T_\lambda{}^\rho(\mathsf{x})\,T_{\rho\sigma}(\mathsf{x})\,u^\sigma_{\text{Landau}}(\mathsf{x})}}. \tag{X.40}$$

With this choice, which in turn determines the *Landau frame*, the heat current $\mathsf{q}(\mathsf{x})$ vanishes, see Eq. (X.38a), so that the dissipative tensor $\boldsymbol{\tau}(\mathsf{x})$ satisfies the condition

$$u^\mu_{\text{Landau}}(\mathsf{x})\,\tau_{\mu\nu}(\mathsf{x}) = 0 \tag{X.41}$$

and reduces to its “viscous” part $\boldsymbol{\pi}(\mathsf{x})$.

For a fluid without conserved quantum number, the Landau definition of the 4-velocity is the only natural one. However, in the presence of a conserved charge, heat conduction now enters the dissipative part of the associated current $\mathsf{n}(\mathsf{x})$, which conflicts with the intuition gained in the non-relativistic case. This implies that the Landau choice does not lead to a simple behavior in the limit of low velocities.

Remark: Relation (X.40) means that the Landau 4-velocity is at every point x an eigenvector of the energy momentum tensor, and more precisely an eigenvector with a negative value — the negative of the local energy density $\epsilon(\mathsf{x})$. This property allows the determination of $\mathsf{u}_{\text{Landau}}(\mathsf{x})$.

Eventually, one may of course choose to work with a general fluid 4-velocity $\mathsf{u}(\mathsf{x})$, and thus to keep both the diffusive charge current(s) and the heat flux density in the dynamical fields (X.33)–(X.37). We shall come back to that point at the end of Sec. X.4.4.

X.4.3 General equations of motion

By substituting the decompositions (X.33), (X.37) into the generic conservation laws (X.2), (X.7), one can obtain model-independent equations of motion, that do not depend on any assumption on the various dissipative currents.

For that purpose, let us introduce the notation

$$\nabla^\mu(\mathsf{x}) \equiv \Delta^{\mu\nu}(\mathsf{x})\,\mathrm{d}_\nu, \tag{X.42a}$$

(az) C. Eckart, 1902–1973 (ba) Л. Д. Ландау = L. D. Landau, 1908–1968 (bb) Е. М. Лифшиц = E. M. Lifshitz, 1915–1985 where d_ν, $\nu \in \{0,1,2,3\}$ denotes the components of the 4-gradient d—involving covariant derivatives in case a non-Minkowski system of coordinates is being used. In geometric formulation, this definition reads

$$\boldsymbol{\nabla}(\mathsf{x}) \equiv \boldsymbol{\Delta}(\mathsf{x}) \cdot \mathsf{d}. \tag{X.42b}$$

As is most obvious in the local rest frame at point x, in which the timelike component $\nabla^0(\mathsf{x})$ vanishes, $\boldsymbol{\nabla}(\mathsf{x})$ is the projection of the gradient on the space-like 3-space orthogonal to the 4-velocity. Using this 3-gradient, the 4-gradient can be written $\mathsf{d} = -\mathsf{u}(\mathsf{u}\cdot\mathsf{d}) + \boldsymbol{\nabla}$, i.e. in terms of components

$$\mathrm{d}_\mu = -u_\mu(\mathsf{u}\cdot\mathsf{d}) + \nabla_\mu. \tag{X.42c}$$

Recognizing that $\mathsf{u}\cdot\mathsf{d}$ is the derivative with respect to (proper) time in the local rest frame, this decomposition has a clear meaning.

Let us further adopt the Landau definition for the flow 4-velocity,[67] which will be simply denoted by $\mathsf{u}(\mathsf{x})$ without subscript.

The charge conservation equation (X.2) first yields

$$\mathrm{d}_\mu N_a^\mu(\mathsf{x}) = u^\mu(\mathsf{x})\mathrm{d}_\mu \mathcal{n}_a(\mathsf{x}) + \mathcal{n}_a(\mathsf{x})\mathrm{d}_\mu u^\mu(\mathsf{x}) + \mathrm{d}_\mu n_a^\mu(\mathsf{x}) = 0. \tag{X.43a}$$

In turn, the conservation of the energy momentum tensor (X.7), projected perpendicular to resp. along the 4-velocity, gives

$$\Delta^\rho{}_\nu(\mathsf{x})\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]u^\mu(\mathsf{x})\mathrm{d}_\mu u^\rho(\mathsf{x}) + \nabla^\rho(\mathsf{x})\mathcal{P}(\mathsf{x}) + \Delta^\rho{}_\nu(\mathsf{x})\mathrm{d}_\mu\pi^{\mu\nu}(\mathsf{x}) = 0 \tag{X.43b}$$

resp.

$$u_\nu(\mathsf{x})\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = -u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) - \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) + u_\nu(\mathsf{x})\mathrm{d}_\mu\pi^{\mu\nu}(\mathsf{x}) = 0.$$

In the latter equation, one can substitute the rightmost term by

$$u_\nu(\mathsf{x})\mathrm{d}_\mu\pi^{\mu\nu}(\mathsf{x}) = \mathrm{d}_\mu\big[u_\nu(\mathsf{x})\pi^{\mu\nu}(\mathsf{x})\big] - \big[\mathrm{d}_\mu u_\nu(\mathsf{x})\big]\pi^{\mu\nu}(\mathsf{x}) = -\big[\mathrm{d}_\mu u_\nu(\mathsf{x})\big]\pi^{\mu\nu}(\mathsf{x}),$$

where the second equality follows from condition (X.41) with $\tau^{\mu\nu} = \pi^{\mu\nu}$ (since $\mathsf{q} = 0$). Using the identity (X.42c), and again the condition $u_\nu\pi^{\mu\nu} = 0$, this leads to

$$u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) + \pi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) = 0. \tag{X.43c}$$

Equations (X.43a)–(X.43c) represent the relations governing the dynamics of a dissipative fluid in the Landau frame.

Remarks:

∗ The last term on the left hand side of Eq. (X.43c) can be further transformed: since the viscous tensor $\boldsymbol{\pi}$ is symmetric, one has

$$\pi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) = \frac{1}{2}\pi^{\mu\nu}(\mathsf{x})\big[\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) + \nabla_\nu(\mathsf{x})u_\mu(\mathsf{x})\big], \tag{X.44}$$

which may be used to give the equation of motion a more symmetric form.

∗ If one adopts Eckart's choice of four-velocity, the resulting equations of motion differ from those given here—for instance, the third term $\mathsf{d}\cdot\mathsf{n}_a(\mathsf{x})$ in Eq. (X.43a) drops out, since $\mathsf{n}_a(\mathsf{x}) = 0$—, yet they are physically totally equivalent.

[67] This choice of form for $\mathsf{u}(\mathsf{x})$ is often announced as "let us work in the Landau frame", which means that the local rest frame at each point of the fluid is the Landau frame.

Entropy law in a dissipative relativistic fluid

Inserting the thermodynamic relation $\epsilon + \mathcal{P} = Ts + \sum \mu_a n_a$ into the dynamical equation (X.43c) and using $\mathrm{d}\epsilon = T\,\mathrm{d}s + \sum \mu_a\,\mathrm{d}n_a$, one finds

$$T(\mathsf{x})\mathrm{d}_\mu\big[s(\mathsf{x})u^\mu(\mathsf{x})\big] = -\pi^{\mu\nu}(\mathsf{x})\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x}) + \sum \mu_a(\mathsf{x})\mathrm{d}_\mu n_a^\mu(\mathsf{x})$$

or equivalently, using the identity $n_a^\mu \mathrm{d}_\mu = n_a^\mu \nabla_\mu$ that follows from $n_a^\mu u_\mu = 0$,

$$\mathrm{d}_\mu\left[s(\mathsf{x})u^\mu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}n_a^\mu(\mathsf{x})\right] = -\pi^{\mu\nu}(\mathsf{x})\frac{\nabla_\mu(\mathsf{x})u_\nu(\mathsf{x})}{T(\mathsf{x})} - \sum n_a^\mu(\mathsf{x})\nabla_\mu\left[\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\right]. \tag{X.45a}$$

In the first term on the right hand side, one can use relation (X.44). With the decompositions $\pi^{\mu\nu} = \varpi^{\mu\nu} + \Pi\Delta^{\mu\nu}$ [Eq. (X.36d)] and

$$\frac{1}{2}(\nabla_\mu u_\nu + \nabla_\nu u_\mu) = \frac{1}{2}\left[\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\Delta_{\mu\nu}(\boldsymbol{\nabla}\cdot\mathsf{u})\right] + \frac{1}{3}\Delta_{\mu\nu}(\boldsymbol{\nabla}\cdot\mathsf{u}) \equiv \mathsf{S}_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}(\boldsymbol{\nabla}\cdot\mathsf{u}),$$

where the $\mathsf{S}_{\mu\nu}$ are the components of a traceless tensor[(68)]—comparing with Eq. (II.17d), this is the rate-of-shear tensor—, while $\boldsymbol{\nabla}\cdot\mathsf{u}$ is the (spatial) 3-divergence of the 4-velocity field, one finds

$$\mathrm{d}_\mu\left[s(\mathsf{x})u^\mu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}n_a^\mu(\mathsf{x})\right] = -\frac{\varpi^{\mu\nu}(\mathsf{x})}{T(\mathsf{x})}\mathsf{S}_{\mu\nu}(\mathsf{x}) - \frac{\Pi(\mathsf{x})}{T(\mathsf{x})}\boldsymbol{\nabla}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x}) - \sum n_a^\mu(\mathsf{x})\nabla_\mu\left[\frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}\right]. \tag{X.45b}$$

The left member of this equation is the 4-divergence of the entropy 4-current $\mathsf{S}(\mathsf{x})$, with components $S^\mu(\mathsf{x})$, comprising on the one hand the convective transport of entropy—which is the only contribution present in the perfect-fluid case, see Eq. (X.22)—, and on the other hand a contribution from the dissipative charge currents.

Remark: When working in the Eckart frame associated to a given conserved quantum number, the corresponding dissipative charge current no longer contributes to the entropy 4-current S—which is obvious since n_a vanishes in that frame!—, but the heat 4-current q does. In an arbitrary frame—i.e. using a different choice of fluid 4-velocity and thereby of local rest frame—, the charge currents n_a and heat current q all contribute to S and to the right hand side of Eq. (X.45b).

Let Ω be the 4-volume that represents the space-time trajectory of the fluid between an initial and a final times. Integrating Eq. (X.45b) over Ω while using the same reasoning as in § X.1.1 b, one sees that the left member will yield the change in the total entropy of the fluid between these two times. This entropy variation must be positive to ensure that the second law of thermodynamics holds. Accordingly, one requests that the integrand should be positive: $\mathrm{d}_\mu S^\mu(\mathsf{x}) \geq 0$. This requirement can be used to build models for the dissipative currents.

X.4.4 First order dissipative relativistic fluid dynamics

The decompositions (X.33), (X.37) are purely algebraic and do not imply anything regarding the physics of the fluid. Any such assumption involves two distinct elements: an equation of state, relating the energy density ϵ to the thermodynamic pressure $\mathcal{P}$ and the conserved-charge densities n_a; and a set of *constitutive equations* that model the dissipative effects, i.e. the diffusive charge 4-currents $\mathsf{N}_a(\mathsf{x})$, the heat flux density $\mathsf{q}(\mathsf{x})$ and the dissipative stress tensor $\boldsymbol{\tau}(\mathsf{x})$.

Several approaches are possible to construct such constitutive equations. A first one would be to compute the conserved-charge 4-currents and energy-momentum tensor starting from an underlying microscopic theory, in particular from a kinetic description of the fluid constituents. Alternatively, one can work directly at the "macroscopic" level, using the various constraints applying to such models.

[(68)] In the notation introduced in the remark at the end of § X.4.1, $\mathsf{S}_{\mu\nu} = \nabla_{\langle\mu}u_{\nu\rangle}$.

A first constraint is that the tensorial structure of the various currents should be the correct one: using as building blocks the 4-velocity u, the 4-gradients of the temperature T, the chemical potential μ, and of u, as well as the projector $\boldsymbol{\Delta}$, one writes the possible expressions of the Lorentz-scalar Π, the 4-vectors n_a and q, and the tensor $\boldsymbol{\varpi}$. Another condition is that the second law of thermodynamics should hold, i.e. that when inserting the dissipative currents in Eq. (X.45b), one obtains a 4-divergence of the entropy 4-current that is always positive.

Working like in § X.4.3 in the Landau frame,[69] in which the heat flux density $\mathsf{q}(\mathsf{x})$ vanishes, and assuming a single conserved charge, the simplest—but not the most general—possibility that satisfies all constraints is to request

$$\boxed{\Pi(\mathsf{x}) = -\zeta(\mathsf{x})\nabla^\mu(\mathsf{x})u_\mu(\mathsf{x})} \tag{X.46a}$$

for the dissipative pressure,

$$\boxed{\varpi^{\mu\nu}(\mathsf{x}) = -\eta(\mathsf{x})\left[\nabla^\mu(\mathsf{x})u^\nu(\mathsf{x}) + \nabla^\nu(\mathsf{x})u^\mu(\mathsf{x}) - \frac{2}{3}\Delta^{\mu\nu}(\mathsf{x})\left[\nabla^\rho(\mathsf{x})u_\rho(\mathsf{x})\right]\right] = -2\eta(\mathsf{x})\mathbf{S}^{\mu\nu}(\mathsf{x})} \tag{X.46b}$$

for the components of the shear stress tensor, and

$$\boxed{n^\mu(\mathsf{x}) = -\kappa(\mathsf{x})\left[\frac{n(\mathsf{x})T(\mathsf{x})}{\epsilon(\mathsf{x})+\mathcal{P}(\mathsf{x})}\right]^2\nabla^\mu(\mathsf{x})\left[\frac{\mu(\mathsf{x})}{T(\mathsf{x})}\right]} \tag{X.46c}$$

for the components of the dissipative conserved-charge 4-current. η, ζ, κ are three positive numbers which implicitly depend on the space-time position, inasmuch as they vary with temperature and chemical potential. The first two ones are obviously the shear and bulk viscosity coefficients, respectively, as hinted at by the similarity with the form (III.26f) of the viscous stress tensor of a non-relativistic Newtonian fluid. Accordingly, the equation of motion (X.43b) in which the dissipative stress tensor is substituted by $\pi^{\mu\nu} = \varpi^{\mu\nu} + \Pi\Delta^{\mu\nu}$ with the forms (X.46a), (X.46b) yields a relativistic version of the Navier–Stokes equation.

What is less obvious is that κ in Eq. (X.46c) does correspond to the heat conductivity—which explains why the coefficient in front of the gradient is written in a rather contrived way.

Inserting the dissipative currents (X.46) in the entropy law (X.45b), the latter becomes

$$\mathsf{d}\cdot\mathsf{S}(\mathsf{x}) = \frac{\boldsymbol{\varpi}(\mathsf{x}):\boldsymbol{\varpi}(\mathsf{x})}{2\eta(\mathsf{x})T(\mathsf{x})} + \frac{\Pi(\mathsf{x})^2}{\zeta(\mathsf{x})T(\mathsf{x})} + \left[\frac{\epsilon(\mathsf{x})+\mathcal{P}(\mathsf{x})}{n(\mathsf{x})T(\mathsf{x})}\right]^2\frac{\mathsf{n}(\mathsf{x})^2}{\kappa(\mathsf{x})T(\mathsf{x})}. \tag{X.47}$$

Since $\mathsf{n}(\mathsf{x})$ is space-like and all three transport coefficients η, ζ, κ positive, the right hand side of this equation is positive, as it should.

The constitutive equations (X.46) only involve first order terms in the derivatives of velocity, temperature, or chemical potential. In keeping, the theory constructed with such Ansätze is referred to as *first order dissipative fluid dynamics*—which is the relativistic generalization of the set of laws valid for Newtonian fluids.

This simple analogy with the non-relativistic case, together with the fact that only 3 transport coefficients (for the case with a single conserved charge) are needed, makes the “traditional” formulation of first-order dissipative relativistic fluid dynamics à la Landau presented here — or its variant in the Eckart frame — attractive. However the approaches suffer from an issue that does not affect the non-relativistic counterpart. Indeed, it has been shown that many solutions of the

[69] The corresponding formulae for Π, $\varpi^{\mu\nu}$ and q^μ valid in the Eckart frame, in which n vanishes, can be found e.g. in Sec. 2.4 of Ref. [45].

relativistic Navier–Stokes(–Fourier) equations in the Landau–Lifshitz or Eckart formulations are unstable against small perturbations [46]. Such disturbances will grow exponentially with time, on a microscopic typical time scale. As a result, the velocity of given modes can quickly exceed the speed of light, which is of course unacceptable in a relativistic theory. In addition, gradients also grow quickly, leading to the breakdown of the small-gradient assumption that implicitly underlies the construction of first-order dissipative fluid dynamics.

Violations of causality actually occur for short-wavelength modes, which from a physical point of view should not be described by fluid dynamics since they involve length scales on which the system is not "continuous". As such, the issue is more mathematical than physical. These modes do however play a role in numerical computations, so that there is indeed a problem when one is not working with an analytical solution.

However, it was shown in 2019 [47] that there exist formulations of relativistic first-order dissipative fluid dynamics—still based on the gradients of temperature and chemical potentials,[70] as well as the 4-velocity, as in the non-relativistic case—, using more general classes of reference frames, that are causal and stable against small linear perturbations. This finding also holds in the general-relativistic context [48]. However, the corresponding equations involve more transport coefficients than the Landau or Eckart frame formulations, and the reference frames do not have as simple a physical interpretation.

X.4.5 Second order dissipative relativistic fluid dynamics

To remedy the instability of the usual Landau–Lifshitz or Eckart formulations of first-order dissipative relativistic fluid dynamics—which is especially a problem for numerical implementations, in which rounding errors will quickly propagate if the theory is unstable—, theories going beyond a first-order expansion in gradients were developed.

Coming back to an arbitrary 4-velocity $\mathsf{u}(\mathsf{x})$, the components of the entropy 4-current $\mathsf{S}(\mathsf{x})$ in a first-order dissipative theory read

$$S^\mu(\mathsf{x}) = \frac{\mathcal{P}(\mathsf{x})g^{\mu\nu}(\mathsf{x}) - T^{\mu\nu}(\mathsf{x})}{T(\mathsf{x})}u_\nu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}N_a^\mu(\mathsf{x}), \tag{X.48a}$$

or equivalently

$$S^\mu(\mathsf{x}) = s(\mathsf{x})u^\mu(\mathsf{x}) - \sum \frac{\mu_a(\mathsf{x})}{T(\mathsf{x})}n_a^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}q^\mu(\mathsf{x}) \tag{X.48b}$$

which reduces to the expression between square brackets on the left hand side of Eq. (X.45b) with Landau's choice of 4-velocity.

This entropy 4-current is *linear* in the dissipative 4-currents $\mathsf{n}(\mathsf{x})$ and $\mathsf{q}(\mathsf{x})$. In addition, it is independent of the velocity 3-gradients—encoded in the expansion rate $\boldsymbol{\nabla}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})$ and the rate-of-shear tensor $\mathbf{S}(\mathsf{x})$—, which play a decisive role in dissipation. That is, the form (X.48) can be generalized. A more general form for the entropy 4-current is thus

$$\mathsf{S}(\mathsf{x}) = s(\mathsf{x})\mathsf{u}(\mathsf{x}) - \frac{\mu_N(\mathsf{x})}{T(\mathsf{x})}\mathsf{n}(\mathsf{x}) + \frac{1}{T(\mathsf{x})}\mathsf{q}(\mathsf{x}) + \frac{1}{T(\mathsf{x})}\mathsf{Q}(\mathsf{x}) \tag{X.49a}$$

or equivalently, component-wise,

$$S^\mu(\mathsf{x}) = s(\mathsf{x})u^\mu(\mathsf{x}) - \frac{\mu_N(\mathsf{x})}{T(\mathsf{x})}n^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}q^\mu(\mathsf{x}) + \frac{1}{T(\mathsf{x})}Q^\mu(\mathsf{x}), \tag{X.49b}$$

with $\mathsf{Q}(\mathsf{x})$ a 4-vector, with components $Q^\mu(\mathsf{x})$, that depends on the flow 4-velocity and its gradients—where $\boldsymbol{\nabla}(\mathsf{x})\cdot\mathsf{u}(\mathsf{x})$ and $\mathbf{S}(\mathsf{x})$ are traditionally replaced by $\Pi(\mathsf{x})$ and $\boldsymbol{\varpi}(\mathsf{x})$—and on the dissipative currents:

$$Q^\mu(\mathsf{x}) = Q^\mu\big(\mathsf{u}(\mathsf{x}), \mathsf{n}(\mathsf{x}), \mathsf{q}(\mathsf{x}), \Pi(\mathsf{x}), \boldsymbol{\varpi}(\mathsf{x})\big). \tag{X.49c}$$

[70] A crucial point is precisely that of the definitions of these notions.

In second order dissipative relativistic fluid dynamics with for simplicity a single conserved charge, the most general form for the additional 4-vector $\mathsf{Q}(\mathsf{x})$ contributing to the entropy density is [49, 50, 51]

$$\mathsf{Q}(\mathsf{x}) = \frac{\beta_0(\mathsf{x})\Pi(\mathsf{x})^2 + \beta_1(\mathsf{x})\mathsf{q}_N(\mathsf{x})^2 + \beta_2(\mathsf{x})\boldsymbol{\varpi}(\mathsf{x}):\boldsymbol{\varpi}(\mathsf{x})}{2T(\mathsf{x})}\mathsf{u}(\mathsf{x}) - \frac{\alpha_0(\mathsf{x})}{T(\mathsf{x})}\Pi(\mathsf{x})\mathsf{q}_N(\mathsf{x}) - \frac{\alpha_1(\mathsf{x})}{T(\mathsf{x})}\boldsymbol{\varpi}(\mathsf{x})\cdot\mathsf{q}_N(\mathsf{x}), \tag{X.50a}$$

where

$$\mathsf{q}_N(\mathsf{x}) \equiv \mathsf{q}(\mathsf{x}) - \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{n(\mathsf{x})}\mathsf{n}(\mathsf{x});$$

component-wise, this reads

$$Q^\mu(\mathsf{x}) = \frac{\beta_0(\mathsf{x})\Pi(\mathsf{x})^2 + \beta_1(\mathsf{x})\mathsf{q}_N(\mathsf{x})^2 + \beta_2(\mathsf{x})\varpi_{\nu\rho}(\mathsf{x})\varpi^{\nu\rho}(\mathsf{x})}{2T(\mathsf{x})}u^\mu(\mathsf{x}) - \frac{\alpha_0(\mathsf{x})}{T(\mathsf{x})}\Pi(\mathsf{x})q_N^\mu(\mathsf{x}) - \frac{\alpha_1(\mathsf{x})}{T(\mathsf{x})}\varpi^\mu{}_\rho(\mathsf{x})q_N^\rho(\mathsf{x}). \tag{X.50b}$$

The 4-vector $\mathsf{Q}(\mathsf{x})$ is now quadratic ("of second order") in the dissipative currents—in the wider sense—$\mathsf{q}(\mathsf{x})$, $\mathsf{n}(\mathsf{x})$, $\Pi(\mathsf{x})$ and $\boldsymbol{\varpi}(\mathsf{x})$, and involves 5 additional coefficients depending on temperature and particle-number density, α_0, α_1, β_0, β_1, and β_2.

Substituting this form of $\mathsf{Q}(\mathsf{x})$ in the entropy 4-current (X.49), the simplest way to ensure that its 4-divergence should be positive is to postulate *linear* relationships between the dissipative currents and the gradients of velocity, chemical potential (or rather of $-\mu/T$), and temperature (or rather, $1/T$), as was done in Eqs. (X.46). This recipe yields differential equations for $\Pi(\mathsf{x})$, $\boldsymbol{\varpi}(\mathsf{x})$, $\mathsf{q}_N(\mathsf{x})$, representing 9 coupled scalar equations of motion. These describe the relaxation—with appropriate characteristic time scales τ_Π, $\tau_{\boldsymbol{\varpi}}$, τ_{q_N} respectively proportional to β_0, β_2, β_1, while the involved "time derivative" is that in the local rest frame, $\mathsf{u}\cdot\mathsf{d}$—, of the dissipative currents towards their first-order expressions (X.46).

Adding up the new equations to the usual ones (X.2) and (X.7), the resulting set of equations, known as (Müller[bc]–)Israel[bd]–Stewart[be] theory, is no longer plagued by the issues that affects the relativistic Navier–Stokes–Fourier equations.

Bibliography for Chapter X

- Andersson & Comer [52];
- Landau–Lifshitz [4, 5], Chapter XV, § 133,134 (perfect fluid) and § 136 (dissipative fluid);
- Romatschke [53];
- Weinberg [54], Chapter 2, § 10 (perfect fluid) and § 11 (dissipative fluid).

[bc] I. Müller, born 1936 [bd] W. Israel, 1931–2022 [be] J. M. Stewart, 1943–2016