CHAPTER XI

Flows of relativistic fluids

Throughout this Chapter $c = 1$.

XI.1 Relativistic fluids at rest

... necessitate the presence of a gravitational field.

XI.2 One-dimensional relativistic flows

XI.2.1 Bjorken flow

To describe (part of the) system—often referred to as "fireball"—created in the collision of two heavy nuclei at extremely high energies, Bjorken[(bf)] proposed to treat it as a perfect fluid with a simple velocity field. In a reference frame $\mathcal{R}_0$ ("center-of-momentum frame") in which the total momentum of the colliding nuclei vanishes, and using Minkowski coordinates such that the momenta of the nuclei before their collision lie along the z-direction, the ansatz for the velocity reads [57]

$$\mathsf{v}^z(\mathsf{x}) = \frac{z}{t} \quad \text{for } |z| < t, \qquad \mathsf{v}^x(\mathsf{x}) = \mathsf{v}^y(\mathsf{x}) = 0, \tag{XI.1}$$

independent of the "transverse" coordinates x and y. Accordingly, the Lorentz factor of the local rest frame at point x is $\gamma(\mathsf{x}) = 1/\sqrt{1 - \mathsf{v}^z(\mathsf{x})^2} = t/\sqrt{t^2 - z^2}$, resulting in the 4-velocity field

$$\mathsf{u}^t(\mathsf{x}) = \frac{t}{\sqrt{t^2 - z^2}}, \quad \mathsf{u}^x(\mathsf{x}) = \mathsf{u}^y(\mathsf{x}) = 0, \quad \mathsf{u}^z(\mathsf{x}) = \frac{z}{\sqrt{t^2 - z^2}}. \tag{XI.2}$$

Note that Eq. (XI.1) coincides with the velocity distribution of *non-interacting* particles emitted at time $t = 0$ at $z = 0$ with a velocity along the z-direction.

XI.2.1 a Milne coordinates

A convenient coordinate system to investigate the properties of the flow defined by Eq. (XI.1) consists of the so-called *Milne coordinates*[(bg)]

$$\tau \equiv \sqrt{t^2 - z^2} \quad , \quad \varsigma \equiv \frac{1}{2} \ln \frac{t + z}{t - z}, \tag{XI.3a}$$

called respectively "Bjorken proper time" and "space-time rapidity". Inverting these equations yield the simple relations

$$t = \tau \cosh \varsigma \quad , \quad z = \tau \sinh \varsigma. \tag{XI.3b}$$

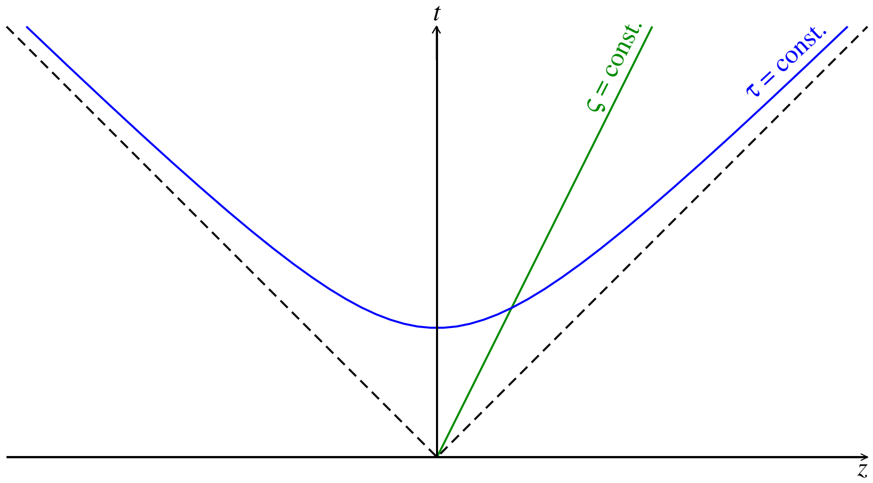

Figure XI.1 – Milne coordinates

Introducing the matrix with entries $\Lambda^{\mu'}{}_{\nu} \equiv \partial x^{\mu'}/\partial x^{\nu}$, where the primed resp. unprimed indices refer to Milne resp. Minkowski coordinates, one quickly finds that the covariant components of a 4-vector V in the two coordinate systems are related by

$$\begin{pmatrix} V^\tau \\ V^\varsigma \end{pmatrix} = \begin{pmatrix} \cosh\varsigma & -\sinh\varsigma \\ -\frac{1}{\tau}\sinh\varsigma & \frac{1}{\tau}\cosh\varsigma \end{pmatrix} \begin{pmatrix} V^t \\ V^z \end{pmatrix}. \tag{XI.4}$$

In particular, this transformation applied to the Bjorken flow 4-velocity (XI.2) yields

$$\mathsf{u}^\tau(\mathsf{x}) = 1, \quad \mathsf{u}^\varsigma(\mathsf{x}) = 0. \tag{XI.5}$$

In turn, the Minkowski components (XI.2) can be rewritten as

$$\mathsf{u}^t(\mathsf{x}) = \cosh\varsigma, \quad \mathsf{u}^z(\mathsf{x}) = \sinh\varsigma \tag{XI.6}$$

which is convenient for calculations.

Since the Milne coordinates (XI.3) are clearly curvilinear, the covariant derivatives d_τ, d_ς do not necessarily coincide with the respective partial derivatives ∂_τ, ∂_ς when acting on vector or more general tensor fields. Instead of working fully in Milne coordinates in the following,[71] we shall compute expressions involving covariant derivatives in Minkowksi coordinates, where $\mathrm{d}_t = \partial_t$ and $\mathrm{d}_z = \partial_z$. Using the chain rule $\partial_{\mu'} = \partial_{\mu'}t\,\partial_t + \partial_{\mu'}z\,\partial_z$ for $\mu' \in \{\tau, \varsigma\}$, one finds

$$\begin{pmatrix} \partial_\tau \\ \frac{1}{\tau}\partial_\varsigma \end{pmatrix} = \begin{pmatrix} \cosh\varsigma & \sinh\varsigma \\ \sinh\varsigma & \cosh\varsigma \end{pmatrix} \begin{pmatrix} \partial_t \\ \partial_z \end{pmatrix} \tag{XI.7}$$

and conversely

$$\begin{pmatrix} \partial_t \\ \partial_z \end{pmatrix} = \begin{pmatrix} \cosh\varsigma & -\sinh\varsigma \\ -\sinh\varsigma & \cosh\varsigma \end{pmatrix} \begin{pmatrix} \partial_\tau \\ \frac{1}{\tau}\partial_\varsigma \end{pmatrix}. \tag{XI.8}$$

From there and the 4-velocity components (XI.6), one arrives at once at the relations

$$u^\mu(\mathsf{x})\partial_\mu = u^t(\mathsf{x})\partial_t + u^z(\mathsf{x})\partial_z = \partial_\tau \qquad \text{and} \qquad \partial_\mu u^\mu(\mathsf{x}) = \partial_t u^t(\mathsf{x}) + \partial_z u^z(\mathsf{x}) = \frac{1}{\tau}. \tag{XI.9}$$

[71] An appendix to this Chapter may be added at some point...

[bf] J. D. Bjorken, born 1934 [bg] E. A. Milne, 1896–1950

Note that $u^\mu\partial_\mu$ coincides with $u^{\mu'}\partial_{\mu'} \equiv u^\tau\partial_\tau + u^\varsigma\partial_\varsigma$, while on the other hand $\partial_\mu u^\mu$ does not equal $\partial_{\mu'}u^{\mu'} \equiv \partial_\tau u^\tau + \partial_\varsigma u^\varsigma$—which trivially vanishes.

Eventually, the projector (X.19b) on the 3-space orthogonal to the flow velocity is readily computed, from which one then deduces the (contravariant) Minkowski components $\nabla^\mu(\mathsf{x}) \equiv \Delta^{\mu\nu}(\mathsf{x})\partial_\nu$ of the 3-gradient (X.42a)

$$\nabla^t = \sinh^2\varsigma\,\partial_t + \cosh\varsigma\sinh\varsigma\,\partial_z = \frac{\sinh\varsigma}{\tau}\partial_\varsigma, \quad \nabla^z = \cosh\varsigma\sinh\varsigma\,\partial_t + \cosh^2\varsigma\,\partial_z = \frac{\cosh\varsigma}{\tau}\partial_\varsigma$$

together with $\nabla^x = \partial_x$, $\nabla^y = \partial_y$, where Eq. (XI.7) was used. Invoking transformation (XI.4), the Milne components of the 3-gradient are

$$\nabla^\tau = 0 \quad , \quad \nabla^\varsigma = \frac{1}{\tau^2}\partial_\varsigma. \tag{XI.10}$$

Consistent with the fact that only u^τ is non-vanishing, ∇^τ vanishes and ∇^ς only involves ∂_ς.

> The reader worried by the appearance of the factor $1/\tau^2$ in ∇^ς will possibly be relieved when realizing that $\nabla_\varsigma \equiv g_{\mu'\varsigma}\nabla^{\mu'} = g_{\varsigma\varsigma}\nabla^\varsigma$—because the metric tensor is still diagonal in Milne coordinates—and that this equals ∂_ς thanks to $g_{\varsigma\varsigma} = \tau^2$.

XI.2.1 b Perfect fluid

For a perfect fluid, with energy-momentum tensor given by Eq. (X.17b), the conservation equation (X.7a) projected parallel resp. orthogonal to the flow 4-velocity leads to the general equations of motion

$$u^\mu(\mathsf{x})\mathrm{d}_\mu\epsilon(\mathsf{x}) + \big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\mathrm{d}_\mu u^\mu(\mathsf{x}) = 0 \tag{XI.11a}$$

resp.

$$\big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]u^\mu(\mathsf{x})\mathrm{d}_\mu u^\rho(\mathsf{x}) + \nabla^\rho(\mathsf{x})\mathcal{P}(\mathsf{x}) = 0, \tag{XI.11b}$$

corresponding to Eqs. (X.43b)–(X.43c) with vanishing viscous tensor.

In the case of the Bjorken flow 4-velocity, for which we derived Eq. (XI.9), these equations become

$$\partial_\tau\epsilon(\mathsf{x}) + \frac{\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})}{\tau} = 0 \tag{XI.12a}$$

and

$$\big[\epsilon(\mathsf{x}) + \mathcal{P}(\mathsf{x})\big]\partial_\tau u^\rho(\mathsf{x}) + \nabla^\rho(\mathsf{x})\mathcal{P}(\mathsf{x}) = 0. \tag{XI.12b}$$

The second of these equations holds in any coordinate system, in particular with Milne coordinates. In the latter, we have found that the components u^τ, u^ς of the velocity are constant, see Eq. (XI.5), in particular independent of τ. That is, the first term on the left hand side of Eq. (XI.12b) vanishes, leaving only

$$\nabla^{\rho'}(\mathsf{x})\mathcal{P}(\mathsf{x}) = 0 \quad \text{for } \rho' \in \{\tau, x, y, \varsigma\}.$$

From Eq. (XI.10), the component $\rho' = \tau$ of this equation is trivial since $\nabla^\tau = 0$. In turn the spatial components read $\partial_x\mathcal{P} = \partial_y\mathcal{P} = 0$, which were obvious from the start since the problem was assumed to be independent of x and y, and

$$\partial_\varsigma\mathcal{P}(\mathsf{x}) = 0. \tag{XI.13}$$

That is, the pressure—and invoking an equation of state, the energy density— is also independent of rapidity.

Coming back to the first equation of motion (XI.12), it can also be rewritten in the form

$$\partial_\tau\big[\tau\epsilon(\mathsf{x})\big] = -\mathcal{P}(\mathsf{x}), \tag{XI.14}$$

which shows that it is the energy-balance equation: the change in the total energy (per unit transverse surface) of a comoving volume[72] is due to the work of pressure forces.

[72] $\mathrm{d}^4\mathsf{x}' = \tau\,\mathrm{d}\tau\,\mathrm{d}\varsigma\,\mathrm{d}x\,\mathrm{d}y$ grows proportionally to τ.

Remarks:

$*$ In a perfect fluid the entropy is conserved: $\mathrm{d}_\mu[s(\mathsf{x})u^\mu(\mathsf{x})] = 0$, see Eq. (X.22), with s the entropy density. This equation can be recast in the form $u^\mu(\mathsf{x})\,\mathrm{d}_\mu s(\mathsf{x}) = -s(\mathsf{x})\,\mathrm{d}_\mu u^\mu(\mathsf{x})$, which using Eq. (XI.9) becomes

$$\partial_\tau s(\mathsf{x}) = -\frac{s(\mathsf{x})}{\tau}. \tag{XI.15}$$

This equation leads at once to $s(\mathsf{x}) \propto 1/\tau$, with the simple interpretation that the total entropy in a comoving fluid volume, proportional to $\tau s(\mathsf{x})$, remains constant in the evolution.

$*$ Ditto for conserved charges: the conservation equation $\mathrm{d}_\mu N_a^\mu(\mathsf{x}) = 0$ [Eq. (X.2)] together with the constitutive relation $N_a^\mu(\mathsf{x}) = n_a(\mathsf{x})u^\mu(\mathsf{x})$ [Eq. (X.17a)] of perfect fluids result in

$$\partial_\tau n_a(\mathsf{x}) = -\frac{n_a(\mathsf{x})}{\tau}. \tag{XI.16}$$

$*$ Bjorken's ansatz (XI.1) for the flow velocity means that an observer $\mathcal{O}_v$ comoving with the fluid at a given point—being say at time t_0 at position z_0 with velocity $v = \mathsf{v}^z(t_0, z_0) = z_0/t_0$—actually moves with constant velocity v with respect to the reference frame $\mathcal{R}_0$. If $\mathcal{R}_0$ is inertial, then $\mathcal{O}_v$ defines another inertial frame $\mathcal{R}_v$: systems of Minkowski coordinates (with parallel-oriented axes) in the two frames are related by a Lorentz boost along the z-direction with velocity v. Instead of v, such a boost is often characterized by its rapidity $\xi \equiv \operatorname{artanh} v = \frac{1}{2}\ln\frac{1+v}{1-v}$. One sees that the boost rapidity ξ is precisely the space-time rapidity ς of the point at which $\mathcal{O}_v$ is sitting. In turn, the statements that the fluid velocity is independent of ς [Eq.(XI.5)] and that this also holds for the locally-measured thermodynamic quantities [(XI.13)] means that all comoving observer $\mathcal{O}_v$, irrespective of their velocity v, view the flow in the same way. The Bjorken flow is thus said to be (longitudinally) boost invariant.

By assuming a simple equation of state, one can derive further results. Let us thus assume that the pressure and energy density are proportional to each other, with a constant—i.e. time- and position-independent—proportionality factor:

$$\mathcal{P}(\mathsf{x}) = c_s^2\epsilon(\mathsf{x}). \tag{XI.17}$$

For instance, $\mathcal{P} = \epsilon/3$ for an ideal gas of ultrarelativistic particles without conserved charge (see Appendix X.C). The notation c_s^2 is not arbitrary but corresponds to the fact that c_s is indeed the (phase) velocity of sound waves in the fluid.

With this equation of state, Eq. (XI.12a) leads at once to

$$\epsilon(\mathsf{x}) \propto \frac{1}{\tau^{1+c_s^2}}, \tag{XI.18}$$

i.e. $\epsilon(\mathsf{x}) \propto 1/\tau^{4/3}$ for an ideal ultrarelativistic gas. That is, the energy density decreases faster than the entropy density—due to the work exerted by pressure.

If one now combines the equation of state (XI.17), the Gibbs–Duhem equation (in absence of conserved charge) $\mathrm{d}\mathcal{P} = s\,\mathrm{d}T$, and the fundamental relation $\epsilon = Ts - \mathcal{P}$, one finds

$$\mathrm{d}\mathcal{P} = c_s^2\,\mathrm{d}\epsilon = \frac{\epsilon + \mathcal{P}}{T}\,\mathrm{d}T.$$

Rewriting the numerator of the rightmost term as $(1 + c_s^2)\epsilon$, there comes

$$\frac{\mathrm{d}\epsilon}{\epsilon} = \frac{1 + c_s^2}{c_s^2}\frac{\mathrm{d}T}{T}.$$

This yields $\epsilon \propto T^{1+c_s^{-2}}$, which together with relation (XI.18) gives

$$T(\mathsf{x}) \propto \frac{1}{\tau^{c_s^2}}, \tag{XI.19}$$

i.e. $T(\mathsf{x}) \propto 1/\tau^{1/3}$ for an ideal ultrarelativistic gas. Since we found earlier that the energy density of such a system decreases as $\tau^{-4/3}$, the behavior of temperature is consistent with the thermal equation of state $\varepsilon \propto T^4$ (and with $s \propto T^3$).

XI.2.1 c First-order dissipative fluid

to be added soon