Chapter X

Fundamental equations of relativistic fluid dynamics

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Under a number of "extreme conditions"—for instance inside compact astrophysical objects, in the early Universe, or in high-energy collisions of heavy nuclei[59]—the "atoms" constituting a fluid can acquire very high kinetic energies, that become comparable to their (rest) mass energy. A non-relativistic description of the medium is then no longer appropriate, and must be replaced by a relativistic model. Some introductory elements of such a description are presented in this Chapter—in which the basic laws governing the dynamics of relativistic fluids are formulated and discussed—, and the following one—which will deal with a few simple analytically tractable solutions of the equations of motions.

As in the non-relativistic case, the basic equations governing the motion of a fluid in the relativistic regime are nothing but formulations of the most fundamental laws of physics, namely conservation laws for "particle number"—in fact, for the various conserved quantum numbers carried by the fluid atoms—, and for energy and momentum (Sec. X.1).

Precisely because the equations simply express general conservation laws, they are not very specific, and contain at first too many degrees of freedom to be tractable. To make progress, it is necessary to introduce models for the fluid under consideration, leading for instance to distinguishing between perfect and dissipative fluids. A convenient way to specify the constitutive equations characteristic of such models is to do so in terms of a fluid four-velocity, which generalizes the non-relativistic flow velocity, yet in a non-unique way (Sec. X.2).

[59] ... as performed at the CERN Large Hadron Collider (LHC) or at the dedicated Relativistic Heavy Ion Collider (RHIC) at the Brookhaven National Laboratory.

Such a fluid four-velocity also automatically singles out a particular reference frame, the local rest frame, in which the conserved currents describing the physics of the fluid take a simpler form, whose physical interpretation is clearer. The perfect fluids are thus those whose properties at each point are spatially isotropic in the corresponding local rest frame, from which there follows that the conserved currents can only depend on the flow four-velocity, not on its derivatives (Sec. X.3). Conversely, when the conserved currents involve (spatial) gradients of the fluid four-velocity, these derivatives signal real fluids with dissipative effects (Sec. X.4).

Two topics that lie beyond the main stream of this Chapter are given in appendices, namely the expression of the conserved currents of relativistic fluid dynamics in terms of underlying microscopic quantities (Sec. X.A) and a discussion of relativistic kinematics (Sec. X.B).

Throughout this Chapter and the following one, the fluids occupy domains of the four-dimensional Minkowski[ax] space-time $\mathscr{M}_4$ of Special Relativity. The position of a generic point of $\mathscr{M}_4$ will be designated by a 4-vector x. Given a reference frame $\mathcal{R}$ and a system of coordinates, those of x will be denoted by $\{x^\mu\} \equiv (x^0, x^1, x^2, x^3)$—where in the case of Minkowski coordinates[60] $x^0 = ct$ with t the time measured by an observer at rest in $\mathcal{R}$.

For the metric tensor $\mathbf{g}$ on $\mathscr{M}_4$, we use the "mostly plus" convention, with signature $(-,+,+,+)$, i.e. in the case of Minkowski coordinates $x_0 = -x^0$ while $x_i = x^i$ for $i = 1,2,3$. Thus, time-like resp. space-like 4-vectors have a negative resp. positive semi-norm.

X.1 Conservation laws

As stated in the introduction, the equations governing the dynamics of fluids in the relativistic regime, just as in the non-relativistic case, embody conservation principles—or more generally, balance equations. More precisely, the usual fundamental equations of relativistic fluid dynamics are differential formulations of these laws. Instead of proceeding as in Chap. III, in which the local formulations were derived from integral ones, we shall hereafter postulate the differential laws, and check or argue that they lead to the expected macroscopic behavior.

> Starting from the local level is more natural here, since one of the tenets underlying relativistic theories, as e.g. quantum field theory, is precisely locality—the absence of action at distance—besides causality. Thus, both conservation equations (X.2) and (X.7) actually emerge as those expressing the invariance of microscopic theories under specific transformations, involving associated Noether currents.

We first discuss the conservation of so-called "charges" (§ X.1.1), then that of energy and momentum, which in a relativistic context are inseparable (§ X.1.2).

X.1.1 Charge conservation

The conservation law that was discussed first in the Chapter III introducing the equations of non-relativistic hydrodynamics was that of mass, which, in the case of a single-component fluid, is fully equivalent to the conservation of particle number. In a relativistic system, the number of particles is strictly speaking not conserved, even if the system is closed. Indeed, thanks to the high kinetic energies available, particle–antiparticle pairs can continuously either be created, or annihilate.

[60] We shall call *Minkowski coordinates* the analog on the space-time $\mathscr{M}_4$ of the Cartesian coordinates on Euclidean space $\mathscr{E}_3$, i.e. those corresponding to a set of four mutually orthogonal 4-vectors $(\mathsf{e}_0, \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3)$ such that the metric tensor has components $g_{\mu\nu} = \mathsf{e}_\mu \cdot \mathsf{e}_\nu = \mathrm{diag}(-1,+1,+1,+1)$ for $\mu, \nu = 0,1,2,3$. They are also alternatively referred to as *Lorentz*[ay] *coordinates*.

[ax] H. Minkowski, 1864–1909 [ay] H. A. Lorentz, 1853–1926

If the particles carry some conserved additive quantum number—as e.g. electric charge or baryon number—, then the value of the quantum number carried by an antiparticle is the negative of that of the corresponding particle. Accordingly, this quantum number is (by definition!) conserved in both creation and annihilation processes.

Throughout this Chapter and the following, such a conserved additive quantum number will be called a "charge", where the reader should beware that this does not necessarily mean the electric charge. Similarly, we shall use the designations "charge (number) density" or "charge flux density". Since there may be several different conserved charges in a given fluid, the corresponding densities will possibly be labeled with an index[(61)] a—which should not be confused with a coordinate index—and accordingly called "charge of type a".

X.1.1 a Local formulation of charge conservation

By definition, the *local charge (number) density* $n_a(t,\vec{r})$ in a fluid is such that the product $n_a(t,\vec{r})\,\mathrm{d}^3\vec{r}$ represents the net amount of charge of type a in the infinitesimal spatial volume $\mathrm{d}^3\vec{r}$ about position $\vec{r}$ at time t.

Since the volume element $\mathrm{d}^3\vec{r}$ depends on the reference frame in which it is measured—remember that in special relativity there is the length contraction phenomenon—, this is also the case of the charge density $n_a(t,\vec{r})$, to ensure that the total charge of type a in $\mathrm{d}^3\vec{r}$ remains independent of the reference frame. Hereafter, $n_a(t,\vec{r})$ will also be denoted by $n_a(\mathsf{x})$.

The *charge flux density* (or *current*) $\vec{j}_{N_a}(t,\vec{r})$ is defined in a similar way, as the amount of charge of type a that crosses a unit area per unit time interval. Since both "unit area" and "unit time interval" in this definition are reference frame-dependent concepts, $\vec{j}_{N_a}(t,\vec{r})$ also depends on the reference frame in which it is measured.

Together, $n_a(\mathsf{x})$ and $\vec{j}_{N_a}(\mathsf{x})$ build up a *charge (number) four-current*[(lxxiii)] $\mathsf{N}_a(\mathsf{x})$, whose contravariant Minkowski components at every point x are $N_a^0(\mathsf{x}) = cn_a(\mathsf{x})$, $N_a^i(\mathsf{x}) = j_{N_a}^i(\mathsf{x})$ for $i = 1,2,3$. This is conveniently summarized in the form

$$\mathsf{N}_a(\mathsf{x}) = \begin{pmatrix} cn_a(\mathsf{x}) \\ \vec{j}_{N_a}(\mathsf{x}) \end{pmatrix} \tag{X.1a}$$

or, somewhat improperly,

$$N_a^\mu(\mathsf{x}) = \begin{pmatrix} cn_a(\mathsf{x}) \\ \vec{j}_{N_a}(\mathsf{x}) \end{pmatrix}. \tag{X.1b}$$

With the help of the charge four-current, the local formulation of the conservation of charge of type a in the motion of the system reads, using coordinates

$$\boxed{\mathrm{d}_\mu N_a^\mu(\mathsf{x}) = 0,} \tag{X.2a}$$

where $\mathrm{d}_\mu \equiv \mathrm{d}/\mathrm{d}x^\mu$ denote the components of the 4-gradient. Denoting the latter, which is a one-form, by d, one may write the even shorter "geometric" (i.e. coordinate-invariant) equation

$$\mathsf{d}\cdot\mathsf{N}_a(\mathsf{x}) = 0, \tag{X.2b}$$

with $\mathsf{d}\,\cdot$ the four-divergence.

Remarks:

∗ Whether $\mathsf{N}_a(\mathsf{x})$ defined by Eq. (X.1) is a 4-vector—that is, whether it behaves as it should under Lorentz transformations—is at first far from clear. That $n_a(\mathsf{x})\,\mathrm{d}^3\vec{r}$ need be a number—i.e. a Lorentz scalar, like $\mathrm{d}^4\mathsf{x} = \mathrm{d}x^0\,\mathrm{d}^3\vec{r}$—suggests that $n_a(\mathsf{x})$ should transform like the time-like component of a

(61) mostly in subscript

(lxxiii) *(Ladungs-) Viererstrom*

4-vector. Yet it is admittedly not clear that the associated spatial part should be the particle flux density.
We shall see in § X.3.3 that assuming that there exists a 4-vector field obeying the conservation equation (X.2) leads in the non-relativistic limit to the above interpretation of its time-like and space-like parts, which may be viewed as a justification.

A better argument is to introduce the particle number 4-current from a microscopic definition, see App. X.A.1. Yet strictly speaking, this goes beyond the fluid-dynamical framework.

$*$ If Minkowski coordinates $\{x^\mu\}$ are used, the components of the 4-gradient d are simply the partial derivatives $\partial_\mu \equiv \partial/\partial x^\mu$, so that Eq. (X.2a) becomes $\partial_\mu N_a^\mu(\mathsf{x}) = 0$.

X.1.1 b Global formulation

Consider in $\mathscr{M}_4$ a space-like 3-dimensional hypersurface Σ—i.e. a hypersurface at every point of which the normal 4-vector is time-like—which extends far enough so that the whole fluid passes through it in its motion; that is, Σ intercepts the worldlines of all fluid particles. Since Σ is 3-dimensional, it may be parameterized (at least in the vicinity of each of its points) by parameters α, β, γ.

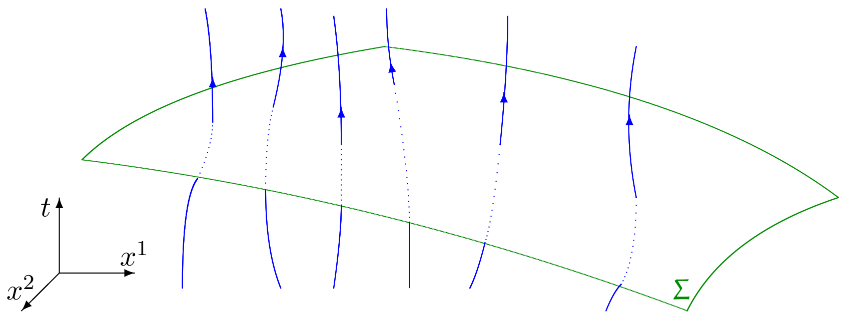

Figure X.1

The total amount N_a of charge of type a in the fluid is the flux of the charge number 4-current $\mathsf{N}_a(\mathsf{x})$ across Σ

$$N_a = \int_\Sigma N_a^\mu(\mathsf{x})\, \mathrm{d}^3\sigma_\mu = \int_\Sigma \mathsf{N}_a(\mathsf{x}) \cdot \mathrm{d}^3\sigma, \tag{X.3}$$

where $\mathrm{d}^3\sigma_\mu$ denotes the components of the 3-hypersurface element

$$\mathrm{d}^3\sigma_\mu \equiv \frac{1}{3!}\sqrt{-\det \mathbf{g}}\, \epsilon_{\mu\nu\rho\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\alpha}\frac{\mathrm{d}x^\rho}{\mathrm{d}\beta}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\gamma}\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma \tag{X.4}$$

with $\epsilon_{\mu\nu\rho\lambda}$ the four-dimensional Levi-Civita symbol, with the convention $\epsilon_{0123} = +1$.[(62)]

Let Ω denote a 4-volume in $\mathscr{M}_4$, and $\partial\Omega$ its 3-surface. Applying the Gauss theorem, the flux of the charge number 4-current across $\partial\Omega$ is the integral of the 4-divergence of $\mathsf{N}_a(\mathsf{x})$ over Ω

$$\oint_{\partial\Omega} \mathsf{N}_a(\mathsf{x}) \cdot \mathrm{d}^3\sigma = \int_\Omega \mathsf{d} \cdot \mathsf{N}_a(\mathsf{x})\, \mathrm{d}^4\mathsf{x}, \tag{X.5}$$

where the right member vanishes thanks to the local expression (X.2) of charge conservation. Splitting $\partial\Omega$ into two space-like parts through which charges enter resp. leave Ω in their motion—the technical criterion is the sign of $\mathsf{N}_a(\mathsf{x}) \cdot \mathrm{d}^3\sigma$—, one finds that there are as many charge carriers that leave as those that enter, which expresses charge conservation globally.

[(62)] This choice is not universal: the alternative convention $\epsilon^{0123} = +1$ results in $\epsilon_{0123} < 0$ due to the odd number of minus signs in the signature of the metric tensor.

X.1.2 Energy-momentum conservation

In a relativistic theory, energy and momentum constitute the temporal and spatial components of a four-vector, the four-momentum. To express the local conservation—in the absence of external forces—of the latter, the densities and flux densities of energy and momentum at each space-time point x must be combined into a four-tensor of degree 2, the *energy-momentum tensor*[(lxxiv)]—also called *stress-energy tensor*—$\mathbf{T}(\mathsf{x})$, of type $\binom{2}{0}$.

This energy-momentum tensor[(63)] may be defined by the physical content of its 16 Minkowski components $T^{\mu\nu}(\mathsf{x})$ in a given reference frame $\mathcal{R}$:

- $T^{00}(\mathsf{x})$ is the energy density;
- $cT^{0j}(\mathsf{x})$ is the j-th component of the energy flux density, with $j = 1,2,3$;
- $\dfrac{1}{c}T^{i0}(\mathsf{x})$ is the density of the i-th component of momentum, with $i = 1,2,3$;
- $T^{ij}(\mathsf{x})$ for $i,j = 1,2,3$ is the momentum flux-density tensor.

(X.6)

All physical quantities are to be measured with respect to the reference frame $\mathcal{R}$.

Remarks:

∗ The similarity of the notations $\mathbf{T}$ resp. $\mathbf{T}$ for the energy-momentum four-tensor resp. the three-dimensional momentum flux-density tensor is not accidental! The former is *the* natural generalization to the 4-dimensional relativistic framework of the latter, just like four-momentum p, with components p^μ is the four-vector associated to the three-dimensional momentum $\vec{p}$. That is, the 3-tensor $\mathbf{T}$ is the spatial part of the 4-tensor $\mathbf{T}$, just like momentum $\vec{p}$ is the spatial part of four-momentum p.

∗ Starting from a microscopic description of the fluid, one can show that the energy-momentum tensor is symmetric, i.e. $T^{\mu\nu}(\mathsf{x}) = T^{\nu\mu}(\mathsf{x})$ for all $\mu,\nu = 0,1,2,3$. Accordingly, only 10 of the 16 entries of $\mathbf{T}$ are independent.

In the absence of external force acting on the fluid, the local conservation of the energy-momentum tensor reads component-wise

$$\boxed{\mathrm{d}_\mu T^{\mu\nu}(\mathsf{x}) = 0 \quad \forall \nu = 0,1,2,3,} \tag{X.7a}$$

which represents four equations: the equation with $\nu = 0$ is the conservation of energy, while the equations $\mathrm{d}_\mu T^{\mu j}(\mathsf{x}) = 0$ for $j = 1,2,3$ are the components of the momentum conservation equation.

In geometric formulation, Eq. (X.7a) becomes

$$\mathrm{d}\cdot\mathbf{T}(\mathsf{x}) = 0. \tag{X.7b}$$

This is exactly the same form as Eq. (X.2b), just like Eqs. (X.2a) and (X.7a) are similar, up to the difference in the tensorial degree of the conserved quantity.

As in § X.1.1 b, one associates to the energy-momentum tensor $\mathbf{T}(\mathsf{x})$ a 4-vector P by

$$\mathsf{P} \equiv \int_\Sigma \mathbf{T}(\mathsf{x})\cdot \mathrm{d}^3\sigma \quad\Leftrightarrow\quad P^\mu = \int_\Sigma T^{\mu\nu}(\mathsf{x})\,\mathrm{d}^3\sigma_\nu, \tag{X.8}$$

with Σ a space-like 3-hypersurface. P represents the total 4-momentum crossing Σ, and invoking the Gauss theorem, Eq. (X.7) implies that it is a conserved quantity.

X.1.3 Fundamental equations in the presence of sources

This may come at some point.

(63) As in the case of the charge number 4-current, the argument showing that $\mathbf{T}(\mathsf{x})$ is a Lorentz tensor is to define it microscopically as a tensor—see App. X.A.2—and to later interpret the physical meaning of the components.

(lxxiv) *Energieimpulstensor*