CHAPTER IX

Convective heat transfer

The examples of dissipative flows we have seen until now were dominated either by viscous effects (Chap. V and § VI.1.4) or by convective motion (Chap. VIII). In either case, the energy-conservation equation (III.36), and in particular the term representing heat conduction, was never playing a dominant role, with the exception of a brief mention of heat transport in the study of static Newtonian fluids (§ V.1.1).

The purpose of this Chapter is to shift the focus, and to discuss motions of Newtonian fluids in which heat is transferred from one region of the fluid to another. A first such type of transfer is heat conduction, which was already encountered in the static case. Under the generic term "convection", or "convective heat transfer", one encompasses flows in which heat is also transported by the moving fluid, not only conductively.

Heat transfer will be caused by differences in temperature in a fluid. Going back to the equations of motion, one can make a few assumptions so as to eliminate or at least suppress other effects, and emphasize the role of temperature gradients in moving fluids (Sec IX.1). A specific instance of fluid motion driven by a temperature difference, yet also controlled by the fluid viscosity, which allows for a richer phenomenology, is then presented in Sec. IX.2.

IX.1 Equations of convective heat transfer

The fundamental equations of the dynamics of Newtonian fluids introduced in Chap. III include heat conduction, in the form of a term involving the gradient of temperature, yet the change in time of temperature does not appear explicitly. To obtain an equation involving the time derivative of temperature, some rewriting of the basic equations is thus needed, which will be done together with a few simplifications (§ IX.1.1). Conduction in a static fluid is then recovered as a limiting case.

In many instances, the main effect of temperature differences is however rather to lead to variations of the mass density, which in turn trigger the fluid motion. To have a more adapted description of such phenomena, a few extra simplifying assumptions are made, leading to a new, closed set of coupled equations (§ IX.1.2).

IX.1.1 Basic equations of heat transfer

Consider a Newtonian fluid submitted to conservative volume forces $\vec{f}_V = -\rho\vec{\nabla}\Phi$. Its motion is governed by the laws established in Chap. III, namely by the continuity equation, the Navier–Stokes equation, and the energy-conservation equation or equivalently the entropy-balance equation, which we now recall.

Expanding the divergence of the mass flux density, the continuity equation (III.9) becomes

$$\frac{\mathrm{D}\rho(t,\vec{r})}{\mathrm{D}t} = -\rho(t,\vec{r})\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}). \tag{IX.1a}$$

In turn, the Navier–Stokes equation (III.30a) may be written in the form

$$\rho(t,\vec{r})\frac{\mathrm{D}\vec{\mathrm{v}}(t,\vec{r})}{\mathrm{D}t} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) - \rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}) + 2\vec{\nabla}\cdot\big[\eta(t,\vec{r})\mathbf{S}(t,\vec{r})\big] + \vec{\nabla}\big[\zeta(t,\vec{r})\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]. \tag{IX.1b}$$

Eventually, straightforward algebra using the continuity equation allows one to rewrite the entropy balance equation (III.41b) as

$$\rho(t,\vec{r})\frac{\mathrm{D}}{\mathrm{D}t}\bigg[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\bigg] = \frac{1}{T(t,\vec{r})}\vec{\nabla}\cdot\big[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\big] + \frac{2\eta(t,\vec{r})}{T(t,\vec{r})}\mathbf{S}(t,\vec{r})\!:\!\mathbf{S}(t,\vec{r}) + \frac{\zeta(t,\vec{r})}{T(t,\vec{r})}\big[\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r})\big]^2. \tag{IX.1c}$$

Since we wish to isolate effects directly related with the transfer of heat, or playing a role in it, we shall make a few assumptions, so as to simplify the above set of equations.

- The transport coefficients η, ζ, κ depend on the local thermodynamic state of the fluid, i.e. on its local mass density ρ and temperature T, and thereby indirectly on time and position. Nevertheless, they will be taken as constant and uniform throughout the fluid, and pulled out of the various derivatives in Eqs. (IX.1b)–(IX.1c). This is a reasonable assumption as long as only small variations of the fluid properties are considered, which is consistent with the next assumption.

  Somewhat abusively, we shall in fact even allow ourselves to consider η resp. κ as uniform in Eq. (IX.1b) resp. (IX.1c), later replace them by related (diffusion) coefficients $\nu = \eta/\rho$ resp. $\alpha = \kappa/\rho c_{\mathcal{P}}$, and then consider the latter as uniform constant quantities.
  The whole procedure is only "justified" in that one can check—by comparing calculations using this assumption with numerical computations performed without the simplifications—that it does not lead to omitting a physical phenomenon.

- The fluid motions under consideration will be assumed to be "slow", i.e. to involve a small flow velocity, in the following sense:
  - The incompressibility condition $\vec{\nabla}\cdot\vec{\mathrm{v}}(t,\vec{r}) = 0$ will hold on the right hand sides of each of Eqs. (IX.1). Accordingly, Eq. (IX.1a) simplifies to $\mathrm{D}\rho(t,\vec{r})/\mathrm{D}t = 0$ while Eq. (IX.1b) becomes the incompressible Navier–Stokes equation

    $$\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r}) = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) - \vec{\nabla}\Phi(t,\vec{r}) + \nu\triangle\vec{\mathrm{v}}(t,\vec{r}), \tag{IX.2}$$

    with a constant and uniform kinematic (shear) viscosity ν.
  - The rate of shear is small, so that its square can be neglected in Eq. (IX.1c). Accordingly, that equation simplifies to

    $$\rho(t,\vec{r})T(t,\vec{r})\frac{\mathrm{D}}{\mathrm{D}t}\bigg[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\bigg] = \kappa\triangle T(t,\vec{r}). \tag{IX.3}$$

The term on the left hand side of that equation can be further rewritten. As a matter of fact, one can show that the differential of the specific entropy in a fluid particle is related to the change in temperature by

$$T\,\mathrm{d}\bigg(\frac{s}{\rho}\bigg) = c_{\mathcal{P}}\,\mathrm{d}T, \tag{IX.4}$$

where $c_{\mathcal{P}}$ denotes the specific heat capacity at constant pressure of the fluid. In turn, this relation translates into an identity relating the material derivatives when the fluid particles are followed in their motion. The left member of Eq. (IX.3) may then be reformulated in terms of the material derivative of the temperature. Introducing the *thermal diffusivity*(lxxii)

$$\alpha \equiv \frac{\kappa}{\rho c_{\mathcal{P}}}, \tag{IX.5}$$

which will from now on be assumed to be constant and uniform in the fluid, where $\rho c_{\mathcal{P}}$ is the volumetric heat capacity at constant pressure, one eventually obtains

$$\boxed{\frac{\mathrm{D}T(t,\vec{r})}{\mathrm{D}t} = \frac{\partial T(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]T(t,\vec{r}) = \alpha\triangle T(t,\vec{r}),} \tag{IX.6}$$

which is sometimes referred to as (*convective*) *heat transfer equation*.

If the fluid is at rest or if its velocity is "small" enough to ensure that the convective term $\vec{\mathsf{v}}\cdot\vec{\nabla}T$ remains negligible, Eq. (IX.6) simplifies to the classical heat diffusion equation, with diffusion constant α.

The thermal diffusivity α thus measures the ability of a medium to transfer heat diffusively, just like the kinematic shear viscosity ν characterizes the diffusive transfer of momentum. Accordingly, both coefficients have the same dimension $\mathsf{L}^2\,\mathsf{T}^{-1}$, and can thus be compared meaningfully. Their relative strength is measured by the dimensionless *Prandtl number*

$$\boxed{\mathrm{Pr} \equiv \frac{\nu}{\alpha} = \frac{\eta c_{\mathcal{P}}}{\kappa}} \tag{IX.7}$$

which in contrast to the Mach, Reynolds, Froude, Ekman, Rossby... numbers encountered in the previous chapters is entirely determined by the fluid, independent of any flow characteristics.

IX.1.2 Boussinesq approximation

If there is a temperature gradient in a fluid, it will lead to a heat flux density, and thereby to the transfer of heat, thus influencing the fluid motion. However, heat exchanges by conduction are often slow—except in metals—, so that another effect due to temperature differences is often the first to play a significant role, namely thermal expansion (or contraction), which will lead to buoyancy (§ IV.1.1) when a fluid particle acquires a mass density different from that of its surroundings.

The simplest approach to account for this effect, due to Boussinesq,(55) consists in considering that even though the fluid mass density changes, nevertheless the motion can to a very good approximation be viewed as incompressible—which is what was assumed in § IX.1.1:

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) \simeq 0, \tag{IX.8}$$

where $\simeq$ is used to allow for small relative variations in the mass density, which are directly related to the expansion rate through Eq. (IX.1a).

Denoting by T_0 a typical temperature in the fluid and ρ_0 the corresponding mass density (strictly speaking, at a given pressure), the effect of thermal expansion on the latter reads

$$\rho(\Theta) = \rho_0(1-\alpha_{(\mathcal{V})}\Theta), \tag{IX.9}$$

with

$$\Theta \equiv T - T_0 \tag{IX.10}$$

the temperature difference measured with respect to the reference value, and

$$\alpha_{(\mathcal{V})} \equiv -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_{\mathcal{P},N} \tag{IX.11}$$

the *thermal expansion coefficient* for volume, where the derivative is taken at the thermodynamic

(55) Hence its denomination *Boussinesq approximation* (for buoyancy).

(lxxii) *Temperaturleitfähigkeit*

point corresponding to the reference value ρ_0. Strictly speaking, the linear regime (IX.9)[56] only holds when $|\alpha_{(\mathcal{V})}\Theta| \ll 1$, as will be assumed hereafter.

Consistent with relation (IX.9), the pressure term in the incompressible Navier–Stokes equation can be approximated as

$$-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) \simeq -\frac{\vec{\nabla}\mathcal{P}(t,\vec{r})}{\rho_0}\big[1+\alpha_{(\mathcal{V})}\Theta(t,\vec{r})\big].$$

Introducing an effective pressure $\mathcal{P}_{\text{eff}}$ that accounts for the leading effect of the potential Φ from which the volume forces derive,

$$\mathcal{P}_{\text{eff.}}(t,\vec{r}) \equiv \mathcal{P}(t,\vec{r}) + \rho_0\Phi(t,\vec{r}),$$

one finds

$$-\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}) - \vec{\nabla}\Phi(t,\vec{r}) \simeq -\frac{\vec{\nabla}\mathcal{P}_{\text{eff.}}(t,\vec{r})}{\rho_0} + \alpha_{(\mathcal{V})}\Theta(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}),$$

where a term of subleading order $\alpha_{(\mathcal{V})}\Theta\vec{\nabla}\mathcal{P}_{\text{eff.}}$ has been dropped. To this level of approximation, the incompressible Navier–Stokes equation (IX.2) becomes

$$\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}) = -\frac{\vec{\nabla}\mathcal{P}_{\text{eff.}}(t,\vec{r})}{\rho_0} + \alpha_{(\mathcal{V})}\Theta(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}) + \nu\triangle\vec{\mathsf{v}}(t,\vec{r}). \tag{IX.12}$$

This form of the Navier–Stokes equation emphasizes the role of a finite temperature difference Θ in providing an extra force density which contributes to the buoyancy, supplementing the effective pressure term.

Eventually, definition (IX.10) together with the convective heat transfer equation (IX.6) lead at once to

$$\frac{\partial\Theta(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\Theta(t,\vec{r}) = \alpha\triangle\Theta(t,\vec{r}). \tag{IX.13}$$

The (*Oberbeck*[at]–)*Boussinesq equations* (IX.8), (IX.12), and (IX.13) represent a closed system of five coupled scalar equations for the dynamical fields $\vec{\mathsf{v}}$, Θ—which in turn determines the variation of the mass density—and $\mathcal{P}_{\text{eff.}}$.

(56)... which is in fact the beginning of a Taylor expansion in Θ.

(at)A. Oberbeck, 1849–1900