VIII.3 Statistical description of turbulence

Instead of handling the turbulent part of the motion like a source of momentum or a sink of kinetic energy for the mean flow, another approach consists in considering its dynamics more carefully (§ VIII.3.1). As already discussed in § VIII.1.4, this automatically involves higher-order autocorrelation functions of the fluctuating velocity, which hints at the interest of looking at the general autocorrelation functions, rather than just their values at equal times and equal positions. This more general approach allows on the one hand to determine length scales of relevance for turbulence (§ VIII.3.2), and on the other hand to motivate a statistical theory of (isotropic) turbulence (§ VIII.3.3).

VIII.3.1 Dynamics of the turbulent motion

Starting from the incompressible Navier–Stokes equation (VIII.8) for the "total" flow velocity $\vec{\mathsf{v}}$ and subtracting from it the Reynolds-averaged equation (VIII.9) for the mean flow, one finds the dynamical equation governing the evolution of the turbulent velocity $\vec{\mathsf{v}}'$, namely [for brevity, the $(t,\vec{r})$-dependence of the fields is omitted]

$$\rho\left[\frac{\partial\vec{\mathsf{v}}'}{\partial t}+\big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\vec{\mathsf{v}}'\right]=-\vec{\nabla}\mathcal{P}'+\eta\triangle\vec{\mathsf{v}}'-\rho\big(\vec{\mathsf{v}}'\cdot\vec{\nabla}\big)\overline{\vec{\mathsf{v}}}-\vec{\nabla}\cdot\big(\rho\vec{\mathsf{v}}'\otimes\vec{\mathsf{v}}'-\mathbf{T}_{\mathrm{R}}\big). \tag{VIII.25a}$$

Equivalently, after dividing by ρ and projecting along the x^i-axis of a coordinate system, one may write

$$\frac{\partial\mathsf{v}'^i}{\partial t}+\big(\overline{\vec{\mathsf{v}}}\cdot\vec{\nabla}\big)\mathsf{v}'^i=-\frac{1}{\rho}\frac{\mathrm{d}\mathcal{P}'}{\mathrm{d}x_i}+\nu\triangle\mathsf{v}'^i-\big(\vec{\mathsf{v}}'\cdot\vec{\nabla}\big)\overline{\mathsf{v}^i}-\frac{\mathrm{d}}{\mathrm{d}x^j}\Big(\mathsf{v}'^i\mathsf{v}'^j-\overline{\mathsf{v}'^i\mathsf{v}'^j}\Big). \tag{VIII.25b}$$

One recognizes on the left hand side of those equations the material derivative of the fluctuating velocity following the mean flow, $\overline{\mathrm{D}}\vec{\mathsf{v}}'/\overline{\mathrm{D}}t$.

From the turbulent Navier–Stokes equation (VIII.25), one finds for the average kinetic energy per unit mass of the fluctuating motion $\overline{k'}\equiv\frac{1}{2}\overline{\vec{\mathsf{v}}'^2}$

$$\frac{\overline{\mathrm{D}}\overline{k'}}{\overline{\mathrm{D}}t}=-\sum_{j=1}^{3}\frac{\mathrm{d}}{\mathrm{d}x^j}\left[\frac{1}{\rho}\overline{\mathcal{P}'\mathsf{v}'^j}+\sum_{i=1}^{3}\left(\frac{1}{2}\overline{\mathsf{v}'_i\mathsf{v}'^i\mathsf{v}'^j}-2\nu\overline{\mathsf{v}'_i\mathbf{S}'^{ij}}\right)\right]-\sum_{i,j=1}^{3}\overline{\mathsf{v}'^i\mathsf{v}'^j}\,\overline{\mathbf{S}}_{ij}-2\nu\sum_{i,j=1}^{3}\overline{\mathbf{S}'^{ij}\mathbf{S}'_{ij}} \tag{VIII.26}$$

with $\mathbf{S}'^{ij}\equiv\dfrac{1}{2}\left(\dfrac{\mathrm{d}\mathsf{v}'^i}{\mathrm{d}x_j}+\dfrac{\mathrm{d}\mathsf{v}'^j}{\mathrm{d}x_i}-\dfrac{2}{3}g^{ij}\vec{\nabla}\cdot\vec{\mathsf{v}}'\right)$ the components of the fluctuating rate-of-shear tensor.

- The first term on the right hand side describes a turbulent yet conservative transport—due to pressure, convective transport by the fluctuating flow itself, or diffusive transport due the viscous friction—, mixing the various length scales: the kinetic energy is transported without loss from the large scales, comparable to that of the variations of the mean flow, to the smaller ones. This process is referred to as *energy cascade*.

- The second term describes the "creation" of turbulent kinetic energy, which is actually extracted from the mean flow: it is precisely—up to the sign!—the loss term in Eq. (VIII.16) describing the transport of kinetic energy in the mean flow.

- Eventually, the rightmost term in Eq. (VIII.26) represents the average energy dissipated as heat by the viscous friction forces, and will hereafter be denoted as $\overline{\dot{\mathcal{E}}}_{\mathrm{diss.}}$.

In a statistically homogeneous and stationary turbulent flow, the amount of energy dissipated by viscous friction equals that extracted by turbulence from the mean flow, i.e.

$$-\sum_{i,j=1}^{3}\overline{\mathsf{v}'^i\mathsf{v}'^j}\,\overline{\mathbf{S}}_{ij}=2\nu\sum_{i,j=1}^{3}\overline{\mathbf{S}'^{ij}\mathbf{S}'_{ij}}. \tag{VIII.27}$$

VIII.3.2 Characteristic length scales of turbulence

VIII.3.2 a Two-point autocorrelation function of the turbulent velocity fluctuations

The fluctuations of the turbulent velocity $\vec{\mathsf{v}}'$ are governed by an unknown probability distribution. Instead of knowing the latter, it is equivalent to rely on the (*auto*)*correlation functions*

$$\kappa^{(n)}_{i_1 i_2 \ldots i_n}(t_1,\vec{r}_1;t_2,\vec{r}_2;\ldots;t_n,\vec{r}_n) \equiv \overline{\mathsf{v}'_{i_1}(t_1,\vec{r}_1)\,\mathsf{v}'_{i_2}(t_2,\vec{r}_2)\cdots\mathsf{v}'_{i_n}(t_n,\vec{r}_n)}\,,$$

in which the components of fluctuations at different instants and positions are correlated with each other. Remember that the 1-point averages vanish, Eq. (VIII.4).

The knowledge of *all* n-point autocorrelation functions is equivalent to that of the probability distribution. Yet the simplest—both from the experimental point of view as well as in numerical simulations—of these functions are the two-point autocorrelation functions [32]

$$\kappa^{(2)}_{ij}(t,\vec{r};t',\vec{r}\,') \equiv \overline{\mathsf{v}'_i(t,\vec{r})\,\mathsf{v}'_j(t',\vec{r}\,')}\,, \tag{VIII.28}$$

which will hereafter be considered only at equal times $t'=t$.

In the case of a statistically stationary turbulent flow,[48] the 2-point autocorrelation functions $\kappa^{(2)}_{ij}(t,\vec{r};t',\vec{r}\,')$ only depend on the time difference $t'-t$, which vanishes if both instants are equal, yielding a function of $\vec{r}$, $\vec{r}\,'$ only. If the turbulence is in addition statistically homogeneous[48]—which necessitates that one considers it far from any wall or obstacle, although this does not yet constitute a sufficient condition—, then the 2-point autocorrelation function only depends on the separation $\vec{X}\equiv\vec{r}\,'-\vec{r}$ of the two positions:

$$\kappa_{ij}(\vec{X}) = \overline{\mathsf{v}'_i(t,\vec{r})\,\mathsf{v}'_j(t,\vec{r}+\vec{X})}\,. \tag{VIII.29}$$

If the turbulence is statistically locally isotropic,[48] the tensor κ_{ij} only depends on the distance $X\equiv|\vec{X}|$ between the two points. Such a statistical local isotropy often represents a good assumption for the structure of the turbulent motion on small scales—again, far from the boundaries of the flow—and will be assumed hereafter.

Consider two points at $\vec{r}$ and $\vec{r}+\vec{X}$. Let $\vec{e}_\parallel$ denote a unit vector along $\vec{X}$, $\vec{e}_\perp$ a unit vector in a direction orthogonal to $\vec{e}_\parallel$, and $\vec{e}\,'_\perp$ perpendicular to both $\vec{e}_\parallel$ and $\vec{e}_\perp$. The component $\mathsf{v}'_\parallel$ of the turbulent velocity—at $\vec{r}$ or $\vec{r}+\vec{X}$—along $\vec{e}_\parallel$ is referred to as "longitudinal", those along $\vec{e}_\perp$ or $\vec{e}\,'_\perp$ ($\mathsf{v}'_\perp$, $\mathsf{v}'_{\perp'}$) as "lateral".

The autocorrelation function (VIII.29) can be expressed with the help of the two-point functions $\kappa_\parallel(X)\equiv\overline{\mathsf{v}'_\parallel(t,\vec{r})\,\mathsf{v}'_\parallel(t,\vec{r}+\vec{X})}$, $\kappa_\perp(X)\equiv\overline{\mathsf{v}'_\perp(t,\vec{r})\,\mathsf{v}'_\perp(t,\vec{r}+\vec{X})}$, and $\kappa'_\perp(X)\equiv\overline{\mathsf{v}'_\perp(t,\vec{r})\,\mathsf{v}'_{\perp'}(t,\vec{r}+\vec{X})}$ as

$$\kappa_{ij}(X) = \frac{X_i X_j}{\vec{X}^2}\big[\kappa_\parallel(X)-\kappa_\perp(X)\big] + \kappa_\perp(X)\,\delta_{ij} + \kappa'_\perp(X)\sum_{k=1}^{3}\frac{\epsilon^{ijk}X_k}{X},$$

with $\{X_i\}$ the Cartesian components of $\vec{X}$, where the last term vanishes for statistically space-parity invariant turbulence,[49] which is assumed to be the case from now on.[50]
Multiplying the incompressibility condition $\vec{\nabla}\cdot\vec{\mathsf{v}}'=0$ with v_j and averaging yields

$$\sum_{i=1}^{3}\frac{\partial\kappa_{ij}(X)}{\partial X_i} = 0,$$

resulting in the identity

$$\kappa_\perp(X) = \kappa_\parallel(X) + \frac{X}{2}\frac{\mathrm{d}\kappa_\parallel(X)}{\mathrm{d}X},$$

[48] This means that the probability distribution of the velocity fluctuations $\vec{\mathsf{v}}'$ is stationary (time-independent) resp. homogeneous (position-independent) resp. locally isotropic (the same for all Cartesian components of $\vec{\mathsf{v}}'$).
[49] Invariance under the space-parity operation is sometimes considered to be part of the isotropy, sometimes not...
[50] In presence of a magnetic field—i.e. in the realm of magnetohydrodynamics—, this last term is indeed present.

which means that κ_{ij} can be expressed in terms of the autocorrelation function $\kappa_\parallel$ only.

VIII.3.2 b Microscopic and macroscopic length scales of turbulence

The assumed statistical isotropy gives $\kappa_\parallel(0) = \overline{[\mathsf{v}_\parallel(t,\vec{r})]^2} = \frac{1}{3}\overline{[\vec{\mathsf{v}}'(t,\vec{r})]^2}$: let $f(X)$ be the function such that $\kappa_\parallel(X) \equiv \frac{1}{3}\overline{[\vec{\mathsf{v}}'(t,\vec{r})]^2}\, f(X)$ and that

- $f(0) = 1$;
- the fluctuations of the velocity at points separated by a large distance X are not correlated with another, so that $\kappa_\parallel(X)$ must vanish: $\lim\limits_{X\to\infty} f(X) = 0$.
- In addition, f is assumed to be integrable over $\mathbb{R}_+$, and such that its integral from 0 to $+\infty$ is convergent.

The function f then defines a typical macroscopic length scale, namely that over which f resp. $\kappa_\parallel$ decreases,[51] the *integral scale* or *external scale*[lxix]

$$L_I \equiv \int_0^\infty f(X)\,\mathrm{d}X. \tag{VIII.30}$$

Empirically, this integral scale is found to be comparable to the scale of the variations of the mean flow velocity, i.e. characteristic for the production of turbulence in the flow. For example, in a flow past an obstacle, L_I is of the same order of magnitude as the size of the obstacle.

Assuming—as has been done till now—locally isotropic and space-parity invariant turbulence, the function $f(X)$ is even, so that its Taylor expansion around $X = 0$ defines a microscopic length scale:

$$f(X) \underset{X\to 0}{\simeq} 1 - \frac{1}{2}\left(\frac{X}{\ell_T}\right)^2 + \mathcal{O}(X^4) \quad \text{with} \quad \ell_T^2 \equiv -\frac{1}{f''(0)} > 0. \tag{VIII.31}$$

ℓ_T is the *Taylor microscale*.[lxx][52]
Let $x_\parallel$ denote the coordinate along $\vec{X}$. One finds

$$\ell_T^2 = \overline{\left[\mathsf{v}'_\parallel(t,\vec{r})\right]^2} \Big/ \overline{\left[\mathrm{d}\mathsf{v}'_\parallel(t,\vec{r})/\mathrm{d}x_\parallel\right]^2}, \tag{VIII.32}$$

i.e. ℓ_T is the typical length scale of the gradients of the velocity fluctuations.

Using the definition of f, the Taylor expansion (VIII.31) can be rewritten as

$$\frac{\overline{\mathsf{v}'_\parallel(t,\vec{r})\,\mathsf{v}'_\parallel(t,\vec{r}+\vec{X})}}{\overline{[\mathsf{v}'_\parallel(t,\vec{r})]^2}} \underset{X\to 0}{\simeq} 1 + \frac{1}{2}\frac{\overline{\mathsf{v}'_\parallel(t,\vec{r})\,\partial_\parallel^2\mathsf{v}'_\parallel(t,\vec{r})}}{\overline{[\mathsf{v}'_\parallel(t,\vec{r})]^2}}X^2,$$

where $\partial_\parallel$ denotes the derivative with respect to $x_\parallel$. Invoking the statistical homogeneity of the turbulence, $\overline{[\mathsf{v}'_\parallel(t,\vec{r})]^2}$ is independent of position, thus of $x_\parallel$, which after differentiation leads successively to $\overline{\mathsf{v}'_\parallel(t,\vec{r})\,\partial_\parallel\mathsf{v}'_\parallel(t,\vec{r})} = 0$ and then $\overline{[\partial_\parallel\mathsf{v}'_\parallel(t,\vec{r})]^2} + \overline{\mathsf{v}'_\parallel\,\partial_\parallel^2\mathsf{v}'_\parallel(t,\vec{r})} = 0$, proving relation (VIII.32).

Remark: Even if the Taylor microscale emerges naturally from the formalism, it does not represent the length scale of the smallest eddies in the flow, despite what one could expect.

To find another, physically more relevant microscopic scale, it is necessary to investigate the behavior of the *longitudinal increment*

$$\delta\mathsf{v}'_\parallel(X) \equiv \mathsf{v}'_\parallel(t,\vec{r}+\vec{X}) - \mathsf{v}'_\parallel(t,\vec{r}) \tag{VIII.33}$$

[51] The reader should think of the example $\kappa_\parallel(X) = \kappa_\parallel(0)\,\mathrm{e}^{-X/L_I}$, or at least $\kappa_\parallel(X) \propto \mathrm{e}^{-X/L_I}$ for X large enough compared to a microscopic scale much smaller than L_I.

[52] ... named after the fluid dynamics practitioner G. I. Taylor, not after B. Taylor of the Taylor series.

[lxix] *Integralskala, äußere Skala* [lxx] *Taylor-Mikroskala*

of the velocity fluctuations, which compares the values of the longitudinal component of the latter at different points. According to the definition of the derivative, $\mathrm{d}\mathsf{v}'_\parallel/\mathrm{d}x_\parallel$ is the limit when $X \to 0$ of the ratio $\delta\mathsf{v}'_\parallel(X)/X$. The microscopic *Kolmogorov*[(as)] *length scale* ℓ_K is then defined by

$$\frac{\overline{[\delta\mathsf{v}'_\parallel(\ell_K)]^2}}{\ell_K^2} \equiv \lim_{X\to 0} \frac{\overline{[\delta\mathsf{v}'_\parallel(X)]^2}}{X^2} = \overline{\left[\frac{\mathrm{d}\mathsf{v}'_\parallel(t,\vec{r})}{\mathrm{d}x_\parallel}\right]^2}. \tag{VIII.34}$$

The role of this length scale will be discussed in the following Section, yet it can already be mentioned that it is the typical scale of the smallest turbulent eddies, and thus the pendant to the integral scale L_I.

Remark: Squaring the longitudinal velocity increment (VIII.33) and averaging under consideration of the statistical homogeneity, one finds when invoking Eq. (VIII.31)

$$\frac{\overline{[\delta\mathsf{v}'_\parallel(X)]^2}}{2\overline{[\mathsf{v}'_\parallel(X)]^2}} \underset{X\to 0}{\sim} \frac{1}{2}\left(\frac{X}{\ell_T}\right)^2.$$

On the other hand, experiments or numerical simulations show that the term on the left hand side of this relation equals about 1 when X is larger than the integral scale L_I. That is, the latter and the Taylor microscale can also be recovered from the longitudinal velocity increment.

VIII.3.3 The Kolmogorov theory (K41) of isotropic turbulence

A first successful statistical theory of turbulence was proposed in 1941 by Kolmogorov for statistically locally isotropic turbulent motion, assuming further stationarity, homogeneity and space-parity invariance [33, 34]. This *K41-theory* describes the fluctuations of the velocity increments $\delta\mathsf{v}'_i(X)$, and relies on two assumptions—originally termed *similarity hypotheses* by Kolmogorov:

1st Kolmogorov hypothesis

> *The probability distributions of the turbulent-velocity increments $\delta\mathsf{v}'_i(X)$, $i=1,2,3$, are universal on separation scales X small compared to the integral scale L_I, and are entirely determined by the kinematic viscosity ν of the fluid and by the average energy dissipation rate per unit mass $\overline{\dot{\mathcal{E}}_{\text{diss.}}}$.* (K41-1)

Here "universality" refers to an independence from the precise process which triggers the turbulence.

Considering e.g. the longitudinal increment, this hypothesis gives for the second moment of the probability distribution

$$\overline{[\delta\mathsf{v}'_\parallel(X)]^2} = \sqrt{\nu\overline{\dot{\mathcal{E}}_{\text{diss.}}}}\,\Phi^{(2)}\!\left(\frac{X}{\ell_K}\right) \quad \text{for } X \ll L_I \quad \text{with} \quad \ell_K = \left(\frac{\nu^3}{\overline{\dot{\mathcal{E}}_{\text{diss.}}}}\right)^{1/4} \tag{VIII.35}$$

and $\Phi^{(2)}$ a universal function, irrespective of the flow under study. The factor $\sqrt{\nu\overline{\dot{\mathcal{E}}_{\text{diss.}}}}$ and the form of ℓ_K follow from dimensional considerations—the n-point autocorrelation function involves another function $\Phi^{(n)}$ multiplying a factor $\big(\nu\overline{\dot{\mathcal{E}}_{\text{diss.}}}\big)^{n/4}$.

The hypothesis (K41-1) amounts to assuming that the physics of the fluctuating motion, far from the scale at which turbulence is created, is fully governed by the available energy extracted from the mean flow—which in the stationary regime equals the average energy dissipated by viscous friction in the turbulent motion—and by the amount of friction.

[(as)] А. Н. Колмогоров = A. N. Kolmogorov, 1903–1987

2nd Kolmogorov hypothesis

The probability distributions of the turbulent-velocity increments $\delta\mathsf{v}'_i(X)$, $i=1,2,3$, is independent of the kinematic viscosity ν of the fluid on separation scales X large compared to the microscopic scale ℓ_K. (K41-2)

The idea here is that viscous friction only plays a role at the microscopic scale, while the rest of the turbulent energy cascade is conservative.

The assumption holds for the longitudinal increment (VIII.35) if and only if $\Phi^{(2)}(x) \underset{x\gg 1}{\sim} B^{(2)}x^{2/3}$ with $B^{(2)}$ a universal constant, i.e. if

$$\overline{[\delta\mathsf{v}'_{\parallel}(X)]^2} \sim B^{(2)}\big(\overline{\dot{\mathcal{E}}_{\text{diss.}}}X\big)^{2/3} \quad \text{for } \ell_K \ll X \ll L_I. \tag{VIII.36}$$

The Kolmogorov $2/3$-*law* (VIII.36) does not involve any length scale: this reflects the length-scale "self-similarity" of the conservative energy-cascading process in the *inertial range*[(lxxi)] $\ell_K \ll X \ll L_I$, in which the only relevant parameter is the energy dissipation rate.

The increase of the autocorrelation function $\overline{[\delta\mathsf{v}'_{\parallel}(X)]^2}$ as $X^{2/3}$ is observed both experimentally and in numerical simulations.[(53)]

A further prediction of the K41-theory regards the energy spectrum of the turbulent motion. Let $\tilde{\vec{\mathsf{v}}}'(t,\vec{k})$ denote the spatial Fourier transform of the fluctuating velocity. Up to a factor involving the inverse of the (infinite) volume of the flow, the kinetic energy per unit mass of the turbulent motion component with wave vector equal to $\vec{k}$ up to $\mathrm{d}^3\vec{k}$ is $\frac{1}{2}[\tilde{\vec{\mathsf{v}}}'(t,\vec{k})]^2\,\mathrm{d}^3\vec{k}$. In the case of statistically isotropic turbulence, $\frac{1}{2}[\tilde{\vec{\mathsf{v}}}'(t,\vec{k})]^2\,\mathrm{d}^3\vec{k} = 2\pi k^2[\tilde{\vec{\mathsf{v}}}'(t,\vec{k})]^2\,\mathrm{d}k \equiv S_E(k)\,\mathrm{d}k$ with $S_E(k)$ the kinetic-energy spectral density.

From the $2/3$-law (VIII.36),[(54)] one can then derive the $-5/3$-*law* for the latter, namely

$$S_E(k) = C_K\,\overline{\dot{\mathcal{E}}_{\text{diss.}}}^{2/3}k^{-5/3} \quad \text{for } L_I^{-1} \ll k \ll \ell_K^{-1} = \left(\frac{\bar{\varepsilon}}{\nu^3}\right)^{1/4}, \tag{VIII.37}$$

with C_K a universal constant, the *Kolmogorov constant*, independent from the fluid or the flow geometry, yet depending—like the $-5/3$-law itself—on the space dimensionality. Experimentally[(53)] one finds $C_K \approx 1.45$.

As already mentioned, the laws (VIII.36) and (VIII.37) provide a rather satisfactory description of the results of experiments or numerical simulations. The K41-theory also predicts that the higher-order moments of the probability distribution of the velocity increments should be universal as well—and the reader can easily determine their scaling behavior $\overline{[\delta\mathsf{v}'_{\parallel}(X)]^n} \sim B^{(n)}\big(\overline{\dot{\mathcal{E}}_{\text{diss.}}}X\big)^{n/3}$ in the inertial range using dimensional arguments—, yet this prediction is no longer supported by experiment: the moments do depend on X as power laws, yet not with the predicted exponents.

A deficiency of Kolmogorov's theory is that in his energy cascade, only eddies of similar size interact with each other to transfer the energy from large to small length scales, which is encoded in the self-similarity assumption. In that picture, the distribution of the eddy sizes is statistically stationary.

In contrast, turbulent motion itself tends to deform eddies, by stretching vortices into tubes of smaller cross section, until they become so small that shear viscosity becomes efficient to counteract this process. This behavior somewhat clashes with Kolmogorov's picture.

(53) Examples from experimental results are presented in Ref. [35, Chapter 5].

(54) ... and assuming that $S_E(k)$ behaves properly, i.e. decreases quickly enough, at large k.

(lxxi) *Trägheitsbereich*