CHAPTER VII

Fluid instabilities

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Small perturbation of fluid flow → dispersion relation. Can involve complex numbers (§ VI.1.4). Exponentially increasing solution? Signals instability of the fluid flow! (against linear perturbation)

VII.1 Gravitational instability in a perfect fluid at rest

VII.1.1 Sound waves in a fluid in a uniform gravity field

Consider a perfect fluid in a constant and uniform gravity field $\vec{g}$, which defines the z direction. As seen in § IV.1.2, in a domain over which the fluid mass density is assumed to be constant the Euler equation

$$\rho_0(t,\vec{r})\left\{\frac{\partial\vec{\mathrm{v}}(t,\vec{r})}{\partial t}+\big[\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathrm{v}}(t,\vec{r})\right\}=-\vec{\nabla}\mathcal{P}(t,\vec{r})+\rho(t,\vec{r})\vec{g} \tag{VII.1}$$

and the continuity equation admit the static solution involving Pascal's law (IV.4)

$$\rho(t,\vec{r})=\rho_0 \quad,\quad \mathcal{P}_0(t,\vec{r})=\mathcal{P}^{(0)}-\rho_0 gz \quad,\quad \vec{\mathrm{v}}_0(t,\vec{r})=\vec{0} \tag{VII.2}$$

with $\mathcal{P}_0$ the pressure at some reference altitude $z=0$.

Let us consider a small adiabatic perturbation [cf. Eq. (VI.1)] of that reference state:

$$\rho(t,\vec{r})=\rho_0+\delta\rho(t,\vec{r}) \tag{VII.3a}$$

$$\mathcal{P}(t,\vec{r})=\mathcal{P}_0(t,\vec{r})+\delta\mathcal{P}(t,\vec{r})=\mathcal{P}^{(0)}-\rho_0 gz+\delta\mathcal{P}(t,\vec{r}) \tag{VII.3b}$$

$$\vec{\mathrm{v}}(t,\vec{r})=\vec{\mathrm{v}}_0(t,\vec{r})+\delta\vec{\mathrm{v}}(t,\vec{r})=\delta\vec{\mathrm{v}}(t,\vec{r}) \tag{VII.3c}$$

Inserting these fields into the continuity equation and Euler equation (VII.1) yields dynamical equations for the perturbations $\delta\rho$, $\delta\mathcal{P}$, $\delta\vec{\mathrm{v}}$. Linearizing these equations in the perturbations, one obtains

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t}+\rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r})=0 \tag{VII.4a}$$

as in Eq. (VI.4a) and

$$\rho_0\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t}=\vec{\nabla}\delta\mathcal{P}(t,\vec{r})-\delta\rho(t,\vec{r})\vec{g}. \tag{VII.4b}$$

Using the adiabaticity condition as in § VI.1.1, the small variations of pressure and mass density are related via the speed of sound (VI.5) (estimated at ρ_0):

$$\delta\mathcal{P}(t,\vec{r})=c_s^2\delta\rho(t,\vec{r}). \tag{VII.4c}$$

Combining the time derivative of Eq. (VII.4a) and the divergence of Eq. (VII.4b) under consideration of relation (VII.4c), one finds

$$\frac{\partial^2 \delta\rho(t,\vec{r})}{\partial t} - c_s^2 \triangle \delta\rho(t,\vec{r}) + \vec{g}\cdot\vec{\nabla}\delta\rho(t,\vec{r}) = 0. \tag{VII.5}$$

This is a linear partial differential equation, which reduces to the classical wave equation in the absence of gravitational field.

Inserting in this evolution equation the Fourier ansatz

$$\delta\rho(t,\vec{r}) = \widetilde{\delta\rho}(\omega,\vec{k})\, \mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)} \tag{VII.6}$$

leads at once to the dispersion relation

$$\omega^2 = c_s^2 \vec{k}^2 + \mathrm{i}\vec{g}\cdot\vec{k} \tag{VII.7}$$

for small perturbations. Effect of gravitation ($\mathrm{i}\vec{g}\cdot\vec{k}$) only when wave vector $\vec{k}$ has a component along the direction or $\vec{g}$.

Sound waves in air on Earth: $c_s \approx 340\ \mathrm{m\cdot s^{-1}}$, $g \approx 9.8\ \mathrm{m\cdot s^{-2}}$. The second term on the right hand side of Eq. (VII.7) becomes comparable in absolute value to the first one when $|\vec{k}| \lesssim 10^{-4}\ \mathrm{m^{-1}}$. This corresponds to wavelengths of order 10 km or higher, i.e. to frequencies below 1 Hz. Effect of gravity on usual sound waves is thus negligible.

VII.1.2 Sound wave in a self-gravitating gas

Instead of the setup of the previous paragraph, in which the fluid is in an external gravity field, let us now consider the case of an isolated perfect fluid, whose mass distribution creates a Newtonian gravitational potential Φ according to the Poisson equation

$$\triangle\Phi(t,\vec{r}) = 4\pi G_{\mathrm{N}}\, \rho(t,\vec{r}) \tag{VII.8}$$

with Newton's gravitational constant $G_{\mathrm{N}} \simeq 6.67\times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$. The paramount example is that of an interstellar cloud of mostly hydrogen.

While Eq. (VII.8) describes the influence of mass density on the gravitational potential, they are further coupled together by the Euler equation

$$\rho(t,\vec{r})\left\{\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r})\right\} = -\vec{\nabla}\mathcal{P}(t,\vec{r}) - \rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}). \tag{VII.9}$$

In addition, mass density and velocity field obey the continuity equation. One could invoke a fourth equation, namely an equation of state, to relate the pressure to the mass density, but this will not be needed afterwards.

Instead, assume[(43)] that we take as starting point the stationary and uniform reference state

$$\rho_0(t,\vec{r}) = \rho_0 \quad,\quad \mathcal{P}_0(t,\vec{r}) = \mathcal{P}_0 \quad,\quad \vec{\mathsf{v}}_0(t,\vec{r}) = \vec{0} \quad,\quad \Phi_0(t,\vec{r}) = \Phi_0. \tag{VII.10}$$

As in § VII.1.1, we consider small adiabatic perturbations of the fields:

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}) \tag{VII.11a}$$
$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0(t,\vec{r}) + \delta\mathcal{P}(t,\vec{r}) \tag{VII.11b}$$
$$\vec{\mathsf{v}}(t,\vec{r}) = \delta\vec{\mathsf{v}}(t,\vec{r}) \tag{VII.11c}$$
$$\Phi(t,\vec{r}) = \Phi_0 + \delta\Phi(t,\vec{r}) \tag{VII.11d}$$

with in addition

$$\delta\mathcal{P}(t,\vec{r}) = c_s^2\delta\rho(t,\vec{r}). \tag{VII.12}$$

[(43)]... following Jeans' historical swindle! This assumption is *not* a valid solution of the equations, unless $\rho_0 = 0$.

Invoking the continuity and Euler equation and the Poisson equation (VII.8), the perturbations satisfy the coupled partial differential equations

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \vec{\nabla}\cdot\Big(\big[\rho_0+\delta\rho(t,\vec{r})\big]\delta\vec{\mathrm{v}}(t,\vec{r})\Big) = 0 \tag{VII.13a}$$

$$\big[\rho_0+\delta\rho(t,\vec{r})\big]\bigg\{\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + \big[\delta\vec{\mathrm{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\delta\vec{\mathrm{v}}(t,\vec{r})\bigg\} = -\vec{\nabla}\delta\mathcal{P}(t,\vec{r}) - \big[\rho_0+\delta\rho(t,\vec{r})\big]\vec{\nabla}\delta\Phi(t,\vec{r}) \tag{VII.13b}$$

and

$$\triangle\delta\Phi(t,\vec{r}) = 4\pi G_{\mathrm{N}}\,\delta\rho(t,\vec{r}). \tag{VII.13c}$$

For small perturbations, we can linearize these equations, which thus simplify to the system

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{\mathrm{v}}(t,\vec{r}) = 0 \tag{VII.14a}$$

$$\rho_0\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} = -\vec{\nabla}\delta\mathcal{P}(t,\vec{r}) - \rho_0\vec{\nabla}\delta\Phi(t,\vec{r}) \tag{VII.14b}$$

$$\triangle\delta\Phi(t,\vec{r}) = 4\pi G_{\mathrm{N}}\,\delta\rho(t,\vec{r}). \tag{VII.14c}$$

Combining the divergence of the linearized Euler equation (VII.14b), using Eq. (VII.14c) for the rightmost term, and the time derivative of Eq. (VII.14a), one obtains the partial differential equation

$$\frac{\partial^2\delta\rho(t,\vec{r})}{\partial t} - c_s^2\triangle\delta\rho(t,\vec{r}) - 4\pi G_{\mathrm{N}}\rho_0\,\delta\rho(t,\vec{r}) = 0 \tag{VII.15}$$

governing the evolution of the perturbation $\delta\rho(t,\vec{r})$. The Fourier ansatz $\delta\rho(t,\vec{r}) = \widetilde{\delta\rho}(\omega,\vec{k})\,\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{r}-\omega t)}$ leads to the dispersion relation

$$\omega^2 = c_s^2\vec{k}^2 - 4\pi G_{\mathrm{N}}\rho_0 = c_s^2\big(k^2-k_{\mathrm{J}}^2\big) \tag{VII.16}$$

between the angular frequency ω and the wave vector $\vec{k}$, where in the second identity we have denoted $k \equiv |\vec{k}|$ and introduced the wave number

$$k_{\mathrm{J}} \equiv \frac{\sqrt{4\pi G_{\mathrm{N}}\rho_0}}{c_s}. \tag{VII.17}$$

According to this dispersion relation, (linear) perturbations with wave number $k > k_J$ propagate with the phase velocity

$$c_\varphi = \frac{\omega}{k} = c_s\sqrt{1-k_{\mathrm{J}}^2/k^2}, \tag{VII.18}$$

close to c_s for very small wavelengths $\lambda = 2\pi/k$ and vanishing for $k = k_{\mathrm{J}}$.

On the other hand, in the case of perturbations with a wavelength larger than the *Jeans*[aq] *length* $\lambda_{\mathrm{J}} \equiv 2\pi/k_{\mathrm{J}}$, the dispersion relation (VII.16) yields two modes with purely imaginary (angular) frequencies

$$\omega_\pm(k) = \pm\mathrm{i}c_s\sqrt{k_{\mathrm{J}}^2-k^2}, \tag{VII.19}$$

which do not propagate since the real part of ω vanishes. While the amplitude of the mode with $\omega = \omega_-$ decreases exponentially, that of the mode with $\omega = \omega_+$ will increase.[44] Since there is no physical argument to discard these wildly growing modes, their existence signals the instability of the reference state (VII.10) with respect to large-wavelength perturbations.

For air ($\rho_0 \simeq 1.3\ \mathrm{kg\cdot m^{-3}}$) at 300 K, resulting in a speed of sound $c_s \simeq 350\ \mathrm{m\cdot s^{-1}}$, the Jeans length is of order 6.7×10^4 km, three orders of magnitude larger than the thickness of the Earth atmosphere — in which perturbations are thus stable.

[44] At least, within the limit of validity of the linear analysis performed here.

[aq] J. Jeans, 1877–1946