V.3 Flows at small Reynolds number

This Section deals with incompressible fluid motions at small Reynolds number $\mathrm{Re} \ll 1$, i.e. flows in the regime in which shear viscous effects predominate over those of inertia in the Navier–Stokes equation. Such fluid motions are also referred to as *Stokes flows* or *creeping flows*.[(lii)]

V.3.1 Physical relevance. Equations of motion

Flows of very different nature may exhibit a small Reynolds number (V.12), because the latter combines physical quantities whose value can vary by many orders of magnitude in Nature.[(26)] A few examples of creeping flows are listed hereafter:

- The motion of fluids past microscopic bodies; the small value of the Reynolds number then reflects the smallness of the length scale L_c; for instance:
  - In water ($\eta \approx 10^{-3}$ Pa·s i.e. $\nu \approx 10^{-6}$ m^2·s^{-1}), a bacteria of size $L_c \approx 5$ μm "swims" with velocity $\mathsf{v}_c \approx 10$ μm·s^{-1}, so that $\mathrm{Re} \approx 5 \cdot 10^{-5}$ for the motion of the water past the bacteria: if the bacteria stops propelling itself, the friction exerted by the water brings it immediately to rest.[(27)] Similarly, creeping flows are employed to describe the motion of reptiles in sand—or more precisely, the flow of sand a past an undulating reptile [24].
  - The motion of a fluid past a suspension of small size (Brownian) particles. This will be studied at further length in § V.3.2.

(26) This is mostly true of the characteristic length and velocity scales and of the shear viscosity; in (non-relativistic) fluids, the mass density is always of the same order of magnitude, up to a factor 10^3.

(27) A longer discussion of the motion of bacteria—from a physicist's point of view—, together with the original formulation of the "scallop theorem", can be found in Ref. [23].

(lii) *schleichende Strömungen*

- The slow-velocity motion of geological material: in that case, the small value of v_c and the large shear viscosity compensate the possibly large value of the typical length scale L_c.
  For example, the motion of the Earth's mantle[(28)] with $L_c \approx 100$ km, $\mathsf{v}_c \approx 10^{-5}$ m·s^{-1}, $\rho \approx 5 \cdot 10^3$ kg·m^{-3} and $\eta \approx 10^{22}$ Pa·s corresponds to a Reynolds number $\mathrm{Re} \approx 5 \cdot 10^{-19}$.

Note that the above examples all represent incompressible flows. For the sake of simplicity, we shall also only consider steady motions.

V.3.1 a Stokes equation

Physically, a small Reynolds number means that the influence of inertia is much smaller than that of shear viscosity. That is, the convective term $(\vec{\mathsf{v}} \cdot \vec{\nabla})\vec{\mathsf{v}}$ in the Navier–Stokes equation is negligible with respect to the viscous contribution. Assuming additionally stationarity—which allows us to drop the time variable—and incompressibility, the Navier–Stokes equation (III.31) simplifies to the *Stokes equation*

$$\vec{\nabla}\mathcal{P}(\vec{r}) = \eta\triangle\vec{\mathsf{v}}(\vec{r}) + \vec{f}_V(\vec{r}). \tag{V.17}$$

This constitutes a linearization of the incompressible Navier–Stokes equation.

Using the relation

$$\vec{\nabla} \times \big[\vec{\nabla} \times \vec{c}(\vec{r})\big] = \vec{\nabla}\big[\vec{\nabla} \cdot \vec{c}(\vec{r})\big] - \triangle\vec{c}(\vec{r}) \tag{V.18}$$

valid for any vector field $\vec{c}(\vec{r})$, the incompressibility condition, and the definition of vorticity, the Stokes equation (in the absence of external volume forces) can be rewritten as

$$\vec{\nabla}\mathcal{P}(\vec{r}) = -\eta\vec{\nabla} \times \vec{\omega}(\vec{r}). \tag{V.19}$$

As a result, the pressure satisfies the differential Laplace equation

$$\triangle\mathcal{P}(\vec{r}) = 0. \tag{V.20}$$

In practice, however, this equation is not the most useful one, because the boundary conditions in a flow are mostly given in terms of the flow velocity, in particular at walls or obstacles, not of the pressure.

Taking the curl of Eq. (V.19) and invoking again relation (V.18) remembering that the vorticity vector is itself already a curl, one finds

$$\triangle\vec{\omega}(\vec{r}) = \vec{0}, \tag{V.21}$$

i.e. the vorticity also obeys the Laplace equation. We shall see in Sec. V.5 that the more general dynamical equation governing the dynamics of vorticity in Newtonian fluids indeed yields Eq. (V.21) in the case of stationary flows at small Reynolds number.

V.3.1 b Properties of the solutions of the Stokes equation

Thanks to the linearity of the Stokes equation (V.17), its solutions possess various properties:[(29)]

- Uniqueness of the solution at fixed boundary conditions.
- Additivity of the solutions: if $\vec{\mathsf{v}}_1$ and $\vec{\mathsf{v}}_2$ are solutions of Eq. (V.17) with respective boundary conditions, then the sum $\lambda_1\vec{\mathsf{v}}_1 + \lambda_2\vec{\mathsf{v}}_2$ with real numbers λ_1, λ_2 is also a solution, for a problem with adequate boundary conditions.
  Physically, the multiplying factors should not be too large, to ensure that the Reynolds number of the new problem remains small. The multiplication of the velocity field $\vec{\mathsf{v}}(\vec{r})$ by a constant λ represents a change in the mass flow rate, while the streamlines (I.15) remain unchanged.

(28) From the mass density, the shear viscosity and the typical speed of sound $c_s \approx 5000$ m·s^{-1} of *transverse* waves—i.e. shear waves, that may propagate in a solid, but not in a fluid—, one constructs a characteristic time scale $t_{\rm mantle} = \eta/\rho c_s^2 \approx 3000$ years. For motions with a typical duration $t_c \ll t_{\rm mantle}$, the Earth's mantle behaves like a deformable solid: for instance, with respect to the propagation of sound waves following an earthquake. On the other hand, for motions on a "geological" time scale $t_c \gg t_{\rm mantle}$, the mantle may be modeled as a fluid.

(29) Proofs can be found e.g. in Ref. [2, Chapter 8.2.3].

The dimensionless velocity field $\vec{v}^*$ associated with the two solutions $\vec{v}(\vec{r})$ and $\lambda\vec{v}(\vec{r})$ is the same, provided the differing characteristic velocities v_c resp. λv_c are used. In turn, these define different values of the Reynolds number. For these solutions, $\vec{v}^*$ as given by Eq. (V.13) is thus independent of the parameter Re, and thereby only depends on the variable $\vec{r}^*$: $\vec{v} = v_c \vec{f}(\vec{r}/L_c)$. This also holds for the corresponding dimensionless pressure $\mathcal{P}^*$.

Using dimensional arguments only, the tangential stress is $\eta\partial v_i/\partial x_j \sim \eta v_c/L_c$, so that the friction force on an object of typical linear size[30] L_c is proportional to $\eta v_c L_c$. This result will now be illustrated on an explicit example [cf. Eq. (V.26)], for which the computation can be performed analytically.

V.3.2 Stokes flow past a sphere

Consider a sphere with radius R immersed in a fluid with mass density ρ and shear viscosity η, which far from the sphere flows with uniform velocity $\vec{v}_\infty$, as sketched in Fig. V.4. The goal is to determine the force exerted by the moving fluid on the sphere, which necessitates the calculation of the pressure and flow velocity. Given the geometry of the problem, a system of spherical coordinates (r, θ, φ) centered on the sphere center will be used.

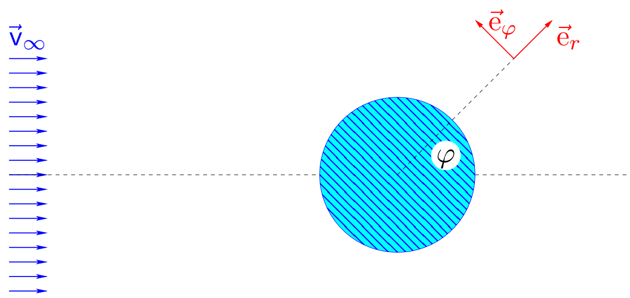

Figure V.4 – Stokes flow past a sphere.

The Reynolds number $\mathrm{Re} = \rho|\vec{v}_\infty|R/\eta$ is assumed to be small, so that the motion in the vicinity of the sphere can be modeled as a creeping flow, which is further taken to be incompressible. For the flow velocity, one looks for a stationary solution of the equations of motion of the form $\vec{v}(\vec{r}) = \vec{v}_\infty + \vec{u}(\vec{r})$, with the boundary condition $\vec{u}(\vec{r}) \to \vec{0}$ when $|\vec{r}| \to \infty$. In addition, the usual impermeability and no-slip conditions hold at the surface of the sphere, resulting in the requirement $\vec{u}(|\vec{r}|=R) = -\vec{v}_\infty$.

Using the linearity of the equations of motion for creeping flows, $\vec{u}$ obeys the equations

$$\triangle\big[\vec{\nabla} \times \vec{u}(\vec{r})\big] = \vec{0}, \tag{V.22a}$$

which reflects Eq. (V.21), and

$$\vec{\nabla} \cdot \vec{u}(\vec{r}) = 0, \tag{V.22b}$$

which comes from the incompressibility condition.

The latter equation is automatically satisfied if $\vec{u}(\vec{r})$ is the curl of some vector field $\vec{V}(\vec{r})$. Using dimensional considerations, the latter should depend linearly on the only explicit velocity scale in the problem, namely $\vec{v}_\infty$. Accordingly, one makes the ansatz[31]

$$\vec{V}(\vec{r}) = \vec{\nabla} \times \big[f(r)\,\vec{v}_\infty\big] = \vec{\nabla}f(r) \times \vec{v}_\infty,$$

[30] As noted in the introduction to Sec. V.2, the characteristic length and velocity scales in a flow are precisely determined by the boundary conditions.

[31] The simpler guesses $\vec{u}(\vec{r}) = f(r)\,\vec{v}_\infty$ or $\vec{u}(\vec{r}) = \vec{\nabla}f(r) \times \vec{v}_\infty$ are both unsatisfactory: the velocity $\vec{u}(\vec{r})$ is then always parallel resp. orthogonal to $\vec{v}_\infty$, so that $\vec{v}(\vec{r})$ cannot vanish everywhere at the surface of the sphere.

with $f(r)$ a function of $r = |\vec{r}|$, i.e. f only depends on the distance from the sphere: apart from the direction of $\vec{\mathsf{v}}_\infty$, which is already accounted for in the ansatz, there is no further preferred spatial direction, so that f should be spherically symmetric.

Relation (V.18) together with the identity $\vec{\nabla}\cdot[f(r)\,\vec{\mathsf{v}}_\infty] = \vec{\nabla}f(r)\cdot\vec{\mathsf{v}}_\infty$ then yield

$$\vec{\mathsf{u}}(\vec{r}) = \vec{\nabla}\times\vec{V}(\vec{r}) = \vec{\nabla}\big[\vec{\nabla}f(r)\cdot\vec{\mathsf{v}}_\infty\big] - \triangle f(r)\,\vec{\mathsf{v}}_\infty. \tag{V.23}$$

The first term on the right hand side has a vanishing curl, and thus does not contribute when inserting $\vec{\mathsf{u}}(\vec{r})$ in equation (V.22a):

$$\vec{\nabla}\times\vec{\mathsf{u}}(\vec{r}) = -\vec{\nabla}\times\big[\triangle f(r)\,\vec{\mathsf{v}}_\infty\big] = -\vec{\nabla}\big[\triangle f(r)\big]\times\vec{\mathsf{v}}_\infty,$$

so that

$$\triangle\Big(\vec{\nabla}\big[\triangle f(r)\big]\Big)\times\vec{\mathsf{v}}_\infty = \vec{0}.$$

Since $f(r)$ does not depend on the azimuthal and polar angles, the vector $\triangle\big(\vec{\nabla}\big[\triangle f(r)\big]\big)$ at position $\vec{r}$ is directed along the radial direction; as thus, it cannot be everywhere parallel to $\vec{\mathsf{v}}_\infty$. Therefore, $\triangle\big(\vec{\nabla}\big[\triangle f(r)\big]\big)$ must vanish identically for the above equation to hold. One can check—for instance using components—the identity $\triangle\big(\vec{\nabla}\big[\triangle f(r)\big]\big) = \vec{\nabla}\big(\triangle[\triangle f(r)]\big)$, so that the equation obeyed by $f(r)$ becomes

$$\triangle[\triangle f(r)] = \text{const.}$$

The integration constant must be zero, since it is a fourth derivative of $f(r)$, while the velocity $\vec{\mathsf{u}}(\vec{r})$, which according to Eq. (V.23) depends on the second derivatives, must vanish as $r\to\infty$. One thus has

$$\triangle[\triangle f(r)] = 0.$$

In spherical coordinates, the Laplacian reads

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2},$$

with ℓ an integer that depends on the angular dependence of the function: given the spherical symmetry of the problem for f, one should take $\ell = 0$. Making the ansatz $\triangle f(r) = C/r^\alpha$, the equation $\triangle[\triangle f(r)] = 0$ is only satisfied for $\alpha = 0$ or 1. Using Eq. (V.23) and the condition $\vec{\mathsf{u}}(\vec{r})\to\vec{0}$ for $r\to\infty$, only $\alpha = 1$ is possible.

The general solution of the linear ordinary differential equation

$$\triangle f(r) = \frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}f(r)}{\mathrm{d}r} = \frac{C}{r} \tag{V.24a}$$

is then given by

$$f(r) = A + \frac{B}{r} + \frac{C}{2}r, \tag{V.24b}$$

where the first two terms in the right member represent the most general of the associated homogeneous equation, while the third term is a particular solution of the inhomogeneous equation.

Equations (V.23) and (V.24) lead to the velocity field

$$\begin{aligned}\vec{\mathsf{u}}(\vec{r}) = \vec{\nabla}\bigg[\bigg(-B\frac{\vec{r}}{r^3} + \frac{C}{2}\frac{\vec{r}}{r}\bigg)\cdot\vec{\mathsf{v}}_\infty\bigg] - \frac{C}{r}\vec{\mathsf{v}}_\infty &= -B\frac{\vec{\mathsf{v}}_\infty - 3\big(\vec{\mathsf{e}}_r\cdot\vec{\mathsf{v}}_\infty\big)\,\vec{\mathsf{e}}_r}{r^3} + \frac{C}{2}\frac{\vec{\mathsf{v}}_\infty - \big(\vec{\mathsf{e}}_r\cdot\vec{\mathsf{v}}_\infty\big)\,\vec{\mathsf{e}}_r}{r} - \frac{C}{r}\vec{\mathsf{v}}_\infty \\ &= -B\frac{\vec{\mathsf{v}}_\infty - 3\big(\vec{\mathsf{e}}_r\cdot\vec{\mathsf{v}}_\infty\big)\,\vec{\mathsf{e}}_r}{r^3} - \frac{C}{2}\frac{\vec{\mathsf{v}}_\infty + \big(\vec{\mathsf{e}}_r\cdot\vec{\mathsf{v}}_\infty\big)\,\vec{\mathsf{e}}_r}{r},\end{aligned}$$

which vanishes at $|\vec{r}|\to\infty$, fulfilling one of the boundary conditions. The other boundary condition $\vec{\mathsf{u}}(|\vec{r}|\!=\!R) = -\vec{\mathsf{v}}_\infty$ at the surface of the sphere translates into

$$\bigg(1 - \frac{B}{R^3} - \frac{C}{2R}\bigg)\vec{\mathsf{v}}_\infty + \bigg(\frac{3B}{R^3} - \frac{C}{2R}\bigg)\big(\vec{\mathsf{e}}_r\cdot\vec{\mathsf{v}}_\infty\big)\,\vec{\mathsf{e}}_r = \vec{0}.$$

This identity must hold irrespective of the orientation of $\vec{e}_r$, which requires both prefactors of $\vec{v}_\infty$ and $\vec{e}_r$ identically vanish, leading to $B = R^3/4$ and $C = 6B/R^2 = 3R/2$ and thereby to

$$\vec{v}(\vec{r}) = \vec{v}_\infty - \frac{3R}{4r}\big[\vec{v}_\infty + (\vec{e}_r\cdot\vec{v}_\infty)\,\vec{e}_r\big] - \frac{R^3}{4r^3}\big[\vec{v}_\infty - 3(\vec{e}_r\cdot\vec{v}_\infty)\,\vec{e}_r\big]. \tag{V.25}$$

Inserting this flow velocity in the Stokes equation (V.17) gives the pressure

$$\mathcal{P}(\vec{r}) = \frac{3}{2}\eta R\frac{\vec{e}_r\cdot\vec{v}_\infty}{r^2} + \text{const},$$

where the unspecified constant is the value of the pressure at infinity, which may be given as an extra boundary condition.

Using $\mathcal{P}(\vec{r})$ and the derivatives)of the velocity field $\vec{v}(\vec{r})$, one can then compute the mechanical stress (III.28) at a point on the surface of the sphere. The total force exerted by the flow on the latter follows from integrating the mechanical stress over the whole surface, and equals

$$\vec{F} = 6\pi R\eta\,\vec{v}_\infty. \tag{V.26}$$

This result is referred as *Stokes' law*. Inverting the point of view, a sphere moving with velocity $\vec{v}_{\text{sphere}}$ in a fluid at rest undergoes a friction force $-6\pi R\eta\vec{v}_{\text{sphere}}$.

Remarks:

∗ For the potential flow of a perfect fluid past a sphere with radius R, the flow velocity is[(32)]

$$\vec{v}(\vec{r}) = \vec{v}_\infty + \frac{R^3}{2r^3}\big[\vec{v}_\infty - 3(\vec{e}_r\cdot\vec{v}_\infty)\,\vec{e}_r\big].$$

That is, the velocity varies much faster in the vicinity of the sphere than for the Stokes flow (V.25): in the latter case, momentum is transported not only convectively but also by viscosity, which redistributes it over a wider region.

The approximation of a flow at small Reynolds number, described by the Stokes equation, actually only holds in the vicinity of the sphere. Far from it, the flow is much less viscous.

∗ In the limit $\eta \to 0$, corresponding to a perfect fluid, the force (V.26) exerted by the flow on the sphere vanishes: this is again the *d'Alembert paradox* encountered in § IV.4.3 c.

∗ The proportionality factor between the sphere velocity and the friction force it experiences is called the *mobility*[(liii)] μ. According to Stokes' law (V.26), for a sphere in the creeping-flow regime one has $\mu = 1/(6\pi R\eta)$.
In his famous article on Brownian motion [25], A. Einstein related this mobility with the diffusion coefficient D of a suspension of small spheres in a fluid at rest:

$$D = \mu k_B T = \frac{k_B T}{6\pi R\eta}.$$

This formula (*Stokes–Einstein equation*) was checked experimentally by J. Perrin, which allowed the latter to determine a value of the Avogadro constant and to prove the "discontinuous structure of matter" [26].

[(32)] The proof can be found e.g. in Landau–Lifshitz [4, 5] § 10 problem 2.

[(liii)] *Beweglichkeit, Mobilität*