VI.3.2 Solitary waves

We now want to go beyond the linear limit considered in § VI.3.1 c for waves at the free surface of a liquid in a gravity field. To that extent, we shall take a few steps back, and first rewrite the dynamical equations of motion and the associated boundary conditions in a dimensionless form (§ VI.3.2 a). This formulation involves two independent parameters, and we shall focus on the limiting case where both are small—yet non-vanishing—and obey a given parametric relation. In that situation, the equation governing the shape of the free surface is the Korteweg–de Vries equation, which in particular describes solitary waves (§ VI.3.2 c).[42]

VI.3.2 a Dimensionless form of the equations of motion

As in § VI.3.1 c, the equations governing the dynamics of gravity waves at the surface of the sea are on the one hand the incompressibility condition

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0, \tag{VI.44a}$$

and on the other hand the Euler equation

$$\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\,\vec{\mathsf{v}}(t,\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(t,\vec{r}) - g\,\vec{e}_z. \tag{VI.44b}$$

The boundary conditions (VI.36) they obey are the absence of vertical velocity at the sea bottom

$$\mathsf{v}_z(t,x,z\!=\!0) = 0, \tag{VI.44c}$$

the identity of the sea vertical velocity with the rate of change of the surface altitude $h_0 + \delta h(t,x)$

$$\mathsf{v}_z\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \frac{\partial\delta h(t,x)}{\partial t} + \mathsf{v}_x(t,\vec{r})\frac{\partial\delta h(t,x)}{\partial x}, \tag{VI.44d}$$

and finally the existence of a uniform pressure at that free surface

$$\mathcal{P}\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \mathcal{P}_0.$$

In the sea at rest, the pressure field is given by the hydrostatic formula

$$\mathcal{P}_{\rm st.}(t,x,z) = \mathcal{P}_0 + \rho g(h_0 - z).$$

Defining the "dynamical pressure" in the sea water as $\mathcal{P}_{\rm dyn.} \equiv \mathcal{P} - \mathcal{P}_{\rm st.}$, the two terms on the right hand side of the Euler equation (VI.44b) can be replaced by $-(1/\rho)\vec{\nabla}\mathcal{P}_{\rm dyn.}$. In addition, the boundary condition at the free surface becomes

$$\mathcal{P}_{\rm dyn.}\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \rho g\delta h(t,x). \tag{VI.44e}$$

Let us now recast Eqs. (VI.44) in a dimensionless form. For that extent, we introduce two characteristic lengths: L_c for long-wavelength motions along x or z, and δh_c for the amplitude of the surface deformation; for durations, we define a scale t_c, which will later be related to L_c with the help of a typical velocity. With these scales, we can construct dimensionless variables

$$t^* \equiv \frac{t}{t_c},\quad x^* \equiv \frac{x}{L_c},\quad z^* \equiv \frac{z}{L_c},$$

and fields:

$$\delta h^* \equiv \frac{\delta h}{\delta h_c},\quad \mathsf{v}_x^* \equiv \frac{\mathsf{v}_x}{\delta h_c/t_c},\quad \mathsf{v}_z^* \equiv \frac{\mathsf{v}_z}{\delta h_c/t_c},\quad \mathcal{P}^* \equiv \frac{\mathcal{P}_{\rm dyn.}}{\rho\,\delta h_c L_c/t_c^2}.$$

Considering the latter as functions of the reduced variables t^*, x^*, z^*, one can rewrite the equations (VI.44a)–(VI.44e). The incompressibility thus becomes

$$\frac{\partial\mathsf{v}_x^*}{\partial x^*} + \frac{\partial\mathsf{v}_z^*}{\partial z^*} = 0, \tag{VI.45a}$$

[42] This Section follows closely the Appendix A of Ref. [28].

and the Euler equation, projected successively on the x and z directions

$$\frac{\partial \mathsf{v}_x^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^* \frac{\partial \mathsf{v}_x^*}{\partial x^*} + \mathsf{v}_z^* \frac{\partial \mathsf{v}_x^*}{\partial z^*}\right) = -\frac{\partial \mathcal{P}^*}{\partial x^*}, \tag{VI.45b}$$

and

$$\frac{\partial \mathsf{v}_z^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^* \frac{\partial \mathsf{v}_z^*}{\partial x^*} + \mathsf{v}_z^* \frac{\partial \mathsf{v}_z^*}{\partial z^*}\right) = -\frac{\partial \mathcal{P}^*}{\partial z^*}, \tag{VI.45c}$$

where we have introduced the dimensionless parameter $\varepsilon \equiv \delta h_c/L_c$. In turn, the various boundary conditions are

$$\mathsf{v}_z^* = 0 \quad \text{at} \quad z^* = 0 \tag{VI.45d}$$

at the sea bottom, and at the free surface

$$\mathsf{v}_z^* = \frac{\partial \delta h^*}{\partial t^*} + \varepsilon \mathsf{v}_x^* \frac{\partial \delta h^*}{\partial x^*} \quad \text{at} \quad z^* = \delta + \varepsilon\,\delta h^* \tag{VI.45e}$$

with $\delta \equiv h_0/L_c$, and

$$\mathcal{P}^* = \frac{g t_c^2}{L_c}\delta h^* \quad \text{at} \quad z^* = \delta + \varepsilon\,\delta h^*.$$

Introducing the further dimensionless number

$$\mathrm{Fr} \equiv \frac{\sqrt{L_c/g}}{t_c},$$

the latter condition becomes

$$\mathcal{P}^* = \frac{1}{\mathrm{Fr}^2}\delta h^* \quad \text{at} \quad z^* = \delta + \varepsilon\,\delta h^*. \tag{VI.45f}$$

Inspecting these equations, one sees that the parameter ε controls the size of nonlinearities—cf. Eqs. (VI.45b), (VI.45c) and (VI.45e)—, while δ measures the depth of the sea in comparison to the typical wavelength L_c. Both parameters are a priori independent: δ is given by the physical setup we want to describe, while ε quantifies the amount of nonlinearity we include in the description.

To make progress, we shall from now on focus on gravity waves on shallow water, i.e. assume $\delta \ll 1$. In addition, we shall only consider small nonlinearities, $\varepsilon \ll 1$. To write down expansions in a consistent manner, we shall assume that the two small parameters are not of the same order, but rather that they obey $\varepsilon \sim \delta^2$. Calculations will be considered up to order $\mathcal{O}(\delta^3)$ or equivalently $\mathcal{O}(\delta\varepsilon)$.

For the sake of brevity, we now drop the superscript * from the dimensionless variables and fields.

VI.3.2 b Velocity potential

If the flow is irrotational, $\partial \mathsf{v}_x/\partial z = \partial \mathsf{v}_z/\partial x$, so that one may transform Eq. (VI.45b) into

$$\frac{\partial \mathsf{v}_x}{\partial t} + \varepsilon\left(\mathsf{v}_x \frac{\partial \mathsf{v}_x}{\partial x} + \mathsf{v}_z \frac{\partial \mathsf{v}_z}{\partial x}\right) + \frac{1}{\mathrm{Fr}^2}\frac{\partial \delta h}{\partial x} = 0. \tag{VI.46}$$

In addition, one may introduce a velocity potential $\varphi(t,x,z)$ such that $\vec{\mathsf{v}} = -\vec{\nabla}\varphi$. With the latter, the incompressibility condition (VI.45a) becomes the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \tag{VI.47}$$

The solution for the velocity potential will be written as an infinite series in z

$$\varphi(t,x,z) = \sum_{n=0}^{\infty} z^n \varphi_n(t,x), \tag{VI.48}$$

with unknown functions $\varphi_n(t,x)$. Substituting this ansatz in the Laplace equation (VI.47) gives after some straightforward algebra

$$\sum_{n=0}^{\infty} z^n \left[\frac{\partial^2\varphi_n(t,x)}{\partial x^2} + (n+1)(n+2)\varphi_{n+2}(t,x)\right] = 0.$$

In order for this identity to hold for arbitrary z—at least, for the values relevant for the flow—, each coefficient should individually vanish, i.e. the φ_n should obey the recursion relation

$$\varphi_{n+2}(t,x) = -\frac{1}{(n+1)(n+2)}\frac{\partial^2\varphi_n(t,x)}{\partial x^2} \quad \text{for} \quad n \in \mathbb{N}. \tag{VI.49}$$

It is thus only necessary to determine φ_0 and φ_1 to know the whole series.

The boundary condition (VI.45d) at the bottom reads $\partial\varphi(t,x,z=0)/\partial z = 0$ for all t and x, which implies $\varphi_1(t,x) = 0$, so that all φ_{2n+1} identically vanish. As a consequence, ansatz (VI.48) with the recursion relation (VI.49) gives

$$\varphi(t,x,z) = \varphi_0(t,x) - \frac{z^2}{2}\frac{\partial^2\varphi_0(t,x)}{\partial x^2} + \frac{z^4}{4!}\frac{\partial^4\varphi_0(t,x)}{\partial x^4} + \dots.$$

Differentiating with respect to x or z yields the components of the velocity $\vec{\mathsf{v}} = -\vec{\nabla}\varphi$

$$\mathsf{v}_x(t,x,z) = -\frac{\partial\varphi_0(t,x)}{\partial x} + \frac{z^2}{2}\frac{\partial^3\varphi_0(t,x)}{\partial x^3} - \frac{z^4}{4!}\frac{\partial^5\varphi_0(t,x)}{\partial x^5} + \dots,$$

$$\mathsf{v}_z(t,x,z) = z\frac{\partial^2\varphi_0(t,x)}{\partial x^2} - \frac{z^3}{3!}\frac{\partial^4\varphi_0(t,x)}{\partial x^4} + \dots.$$

Introducing the notation $\mathsf{u}(t,x) \equiv -\partial\varphi_0(t,x)/\partial x$ and anticipating that the maximal value of z relevant for the problem is of order δ, these components may be expressed as

$$\mathsf{v}_x(t,x,z) = \mathsf{u}(t,x) - \frac{z^2}{2}\frac{\partial^2\mathsf{u}(t,x)}{\partial x^2} + o(\delta^3), \tag{VI.50a}$$

$$\mathsf{v}_z(t,x,z) = -z\frac{\partial\mathsf{u}(t,x)}{\partial x} + \frac{z^3}{3!}\frac{\partial^3\mathsf{u}(t,x)}{\partial x^3} + o(\delta^3), \tag{VI.50b}$$

where the omitted terms are beyond $\mathcal{O}(\delta^3)$.

Linear waves rediscovered

If we momentarily set $\varepsilon = 0$—which amounts to linearizing the equations of motion and boundary conditions—, consistency requires that we consider equations up to order δ at most. That is, we keep only the first terms from Eqs. (VI.50): at the surface at $z \simeq \delta$, they become

$$\mathsf{v}_x(t,x,z=\delta) \simeq \mathsf{u}(t,x), \qquad \mathsf{v}_z(t,x,z=\delta) \simeq -\delta\frac{\partial\mathsf{u}(t,x)}{\partial x}, \tag{VI.51a}$$

while the boundary condition (VI.45e) simplifies to

$$\mathsf{v}_z(t,x,z=\delta) = \frac{\partial\delta h(t,x)}{\partial t} = \delta\frac{\partial\phi(t,x)}{\partial t}, \tag{VI.51b}$$

where we have introduced $\phi(t,x) \equiv \delta h(t,x)/\delta$. Meanwhile, Eq. (VI.46) with $\varepsilon = 0$ reads

$$\frac{\partial\mathsf{v}_x(t,x)}{\partial t} + \frac{\delta}{\mathrm{Fr}^2}\frac{\partial\phi(t,x)}{\partial x} = 0. \tag{VI.51c}$$

Together, Eqs. (VI.51a)–(VI.51c) yield after some straightforward manipulations the equation

$$\frac{\partial^2\mathsf{u}(t,x)}{\partial t^2} - \frac{\delta}{\mathrm{Fr}^2}\frac{\partial^2\mathsf{u}(t,x)}{\partial x^2} = 0, \tag{VI.52}$$

i.e. a linear equation describing waves with the dimensionless phase velocity $\sqrt{\delta}/\mathrm{Fr} = \sqrt{gh_0}/(L_c/t_c)$. Since the scaling factor of x resp. t is L_c resp. t_c, the corresponding dimensionful phase velocity is $c_\varphi = \sqrt{gh_0}$, as was already found in § VI.3.1 c for waves on shallow sea.

Until now, the scaling factor t_c was independent from L_c. Choosing $t_c \equiv L_c/\sqrt{gh_0}$, i.e. the unit in which times are measured, the factor δ/Fr^2 equals 1, leading to the simpler-looking equation

$$\frac{\partial \mathsf{v}_x(t,x,z)}{\partial t} + \varepsilon\left[\mathsf{v}_x(t,x,z)\frac{\partial \mathsf{v}_x(t,x,z)}{\partial x} + \mathsf{v}_z(t,x,z)\frac{\partial \mathsf{v}_z(t,x,z)}{\partial x}\right] + \frac{\partial \phi(t,x)}{\partial x} = 0 \qquad \text{(VI.53)}$$

instead of Eq. (VI.46).

VI.3.2 c Non-linear waves on shallow water

Taking now $\varepsilon \neq 0$ and investigating the equations up to order $\mathcal{O}(\delta^3)$, $\mathcal{O}(\delta\varepsilon)$, Eqs. (VI.50) at the free surface at $z = \delta(1+\varepsilon\phi)$ become

$$\mathsf{v}_x\big(t,x,z{=}\delta(1+\varepsilon\phi)\big) = \mathsf{u}(t,x) - \frac{\delta^2}{2}\frac{\partial^2 \mathsf{u}(t,x)}{\partial x^2}, \qquad \text{(VI.54a)}$$

$$\mathsf{v}_z\big(t,x,z{=}\delta(1+\varepsilon\phi)\big) = -\delta\big[1+\varepsilon\phi(t,x)\big]\frac{\partial \mathsf{u}(t,x)}{\partial x} + \frac{\delta^3}{6}\frac{\partial^3 \mathsf{u}(t,x)}{\partial x^3}. \qquad \text{(VI.54b)}$$

Inserting these velocity components in (VI.53) while retaining only the relevant orders yields

$$\frac{\partial \mathsf{u}(t,x)}{\partial t} - \frac{\delta^2}{2}\frac{\partial^3 \mathsf{u}(t,x)}{\partial t\,\partial x^2} + \varepsilon\,\mathsf{u}(t,x)\frac{\partial \mathsf{u}(t,x)}{\partial x} + \frac{\partial \phi(t,x)}{\partial x} = 0. \qquad \text{(VI.55)}$$

On the other hand, the velocity components are also related by the boundary condition (VI.45e), which reads

$$\mathsf{v}_z\big(t,x,z{=}\delta(1+\varepsilon\phi)\big) = \delta\frac{\partial \phi(t,x)}{\partial t} + \delta\varepsilon\,\mathsf{v}_x\big(t,x,z{=}\delta(1+\varepsilon\phi)\big)\frac{\partial \phi(t,x)}{\partial x}.$$

Substituting Eq. (VI.54a) resp. (VI.54b) in the right resp. left member yields

$$\frac{\partial \phi(t,x)}{\partial t} + \varepsilon\,\mathsf{u}(t,x)\frac{\partial \phi(t,x)}{\partial x} + \big[1+\varepsilon\phi(t,x)\big]\frac{\partial \mathsf{u}(t,x)}{\partial x} - \frac{\delta^2}{6}\frac{\partial^3 \mathsf{u}(t,x)}{\partial x^3} = 0. \qquad \text{(VI.56)}$$

To leading order in δ and ε, the system of nonlinear partial differential equations (VI.55)–(VI.56) simplifies to the linear system

$$\begin{cases} \dfrac{\partial \mathsf{u}(t,x)}{\partial t} + \dfrac{\partial \phi(t,x)}{\partial x} = 0 \\[2mm] \dfrac{\partial \phi(t,x)}{\partial t} + \dfrac{\partial \mathsf{u}(t,x)}{\partial x} = 0, \end{cases}$$

which admits the solution $\mathsf{u}(t,x) = \phi(t,x)$ under the condition

$$\frac{\partial \mathsf{u}(t,x)}{\partial t} + \frac{\partial \mathsf{u}(t,x)}{\partial x} = 0, \qquad \text{(VI.57)}$$

which describes a traveling wave with (dimensionless) velocity 1, $\mathsf{u}(t,x) = \mathsf{u}(x-t)$. We again recover the linear sea surface waves which we have already encountered twice.

Going to next-to-leading order $\mathcal{O}(\delta^2)$, $\mathcal{O}(\varepsilon)$, we look for solutions in the form

$$\mathsf{u}(t,x) = \phi(t,x) + \varepsilon\,\mathsf{u}^{(\varepsilon)}(t,x) + \delta^2\mathsf{u}^{(\delta)}(t,x) \qquad \text{(VI.58)}$$

with ϕ, $\mathsf{u}^{(\varepsilon)}$, $\mathsf{u}^{(\delta)}$ functions that obey condition (VI.57) up to terms of order ε or δ^2. Inserting this ansatz in Eqs. (VI.55)–(VI.56) yields the system

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{\partial \phi}{\partial x} + \varepsilon\dfrac{\partial \mathsf{u}^{(\varepsilon)}}{\partial x} + \delta^2\dfrac{\partial \mathsf{u}^{(\delta)}}{\partial x} + 2\varepsilon\,\phi\dfrac{\partial \phi}{\partial x} - \dfrac{\delta^2}{6}\dfrac{\partial^3 \phi}{\partial x^3} = 0 \\[2mm] \dfrac{\partial \phi}{\partial t} + \dfrac{\partial \phi}{\partial x} + \varepsilon\dfrac{\partial \mathsf{u}^{(\varepsilon)}}{\partial t} + \delta^2\dfrac{\partial \mathsf{u}^{(\delta)}}{\partial t} + \varepsilon\,\phi\dfrac{\partial \phi}{\partial x} - \dfrac{\delta^2}{2}\dfrac{\partial^3 \phi}{\partial x^2\,\partial t} = 0, \end{cases}$$

where for the sake of brevity, the (t, x)-dependence of the functions was not written. Subtracting both equations and using condition (VI.57) to relate the time and space derivatives of ϕ, $\mathsf{u}^{(\varepsilon)}$, and $\mathsf{u}^{(\delta)}$, one finds

$$\varepsilon\left[\frac{\partial \mathsf{u}^{(\varepsilon)}(t,x)}{\partial x}+\frac{1}{2}\phi(t,x)\frac{\partial\phi(t,x)}{\partial x}\right]+\delta^2\left[\frac{\partial \mathsf{u}^{(\delta)}(t,x)}{\partial x}-\frac{1}{3}\frac{\partial^3\phi(t,x)}{\partial x^3}\right]=0.$$

Since the two small parameters ε and δ are independent, each term between square brackets in this identity must identically vanish. Straightforward integrations then yield

$$\mathsf{u}^{(\varepsilon)}(t,x)=-\frac{1}{4}\phi(t,x)+C^{(\varepsilon)}(t),\qquad \mathsf{u}^{(\delta)}(t,x)=\frac{1}{3}\frac{\partial^2\phi(t,x)}{\partial x^2}+C^{(\delta)}(t),$$

with $C^{(\varepsilon)}$, $C^{(\delta)}$ two functions of time only.

These functions can then be substituted in the ansatz (VI.58). Inserting the latter in Eq. (VI.56) yields an equation involving the unknown function ϕ only, namely

$$\frac{\partial\phi(t,x)}{\partial t}+\frac{\partial\phi(t,x)}{\partial x}+\frac{3}{2}\varepsilon\,\phi(t,x)\frac{\partial\phi(t,x)}{\partial x}+\frac{1}{6}\delta^2\frac{\partial^3\phi(t,x)}{\partial x^3}=0. \qquad \text{(VI.59)}$$

The first two terms only are those of the linear-wave equation of motion (VI.57). Since the the nonlinear corrections in ε and δ also obey the same condition, it is fruitful to perform a change of variables from (t, x) to (τ, ξ) with $\tau \equiv t$, $\xi \equiv x - t$. Equation (VI.59) then becomes

$$\boxed{\frac{\partial\phi(\tau,\xi)}{\partial\tau}+\frac{3}{2}\varepsilon\,\phi(\tau,\xi)\frac{\partial\phi(\tau,\xi)}{\partial\xi}+\frac{1}{6}\delta^2\frac{\partial^3\phi(\tau,\xi)}{\partial\xi^3}=0,} \qquad \text{(VI.60)}$$

which is the *Korteweg–de Vries equation*.[ao],[ap]

Remark: By rescaling the variables τ and ξ to a new set $(\mathrm{\tau}, \mathrm{\xi})$, one can actually absorb the parameters ε, δ which were introduced in the derivation. Accordingly, the equation takes the form

$$\boxed{\frac{\partial\phi(\mathrm{\tau},\mathrm{\xi})}{\partial\mathrm{\tau}}+6\phi(\mathrm{\tau},\mathrm{\xi})\frac{\partial\phi(\mathrm{\tau},\mathrm{\xi})}{\partial\mathrm{\xi}}+\frac{\partial^3\phi(\mathrm{\tau},\mathrm{\xi})}{\partial\mathrm{\xi}^3}=0,} \qquad \text{(VI.61)}$$

which is the more standard form of the Korteweg–de Vries equation.

Solitary waves

The Korteweg–de Vries (KdV) equation admits many different solutions. Among those, there is the class of *solitary waves* or *solitons*, which describe signals that propagate without changing their shape.

A specific subclass of solitons of the KdV equation of special interest in fluid dynamics consists of those which at each given instant vanish at (spatial) infinity. As solutions of the normalized equation (VI.61), they read

$$\phi(\mathrm{\tau},\mathrm{\xi})=\frac{\phi_0}{\cosh^2\left[\sqrt{\phi_0/2}\,(\mathrm{\xi}-2\phi_0\mathrm{\tau})\right]} \qquad \text{(VI.62a)}$$

with ϕ_0 the amplitude of the wave. Note that ϕ_0 must be nonnegative, which means that these solutions describe bumps above the mean sea level—which is indeed the only instance of solitary wave observed experimentally on the surface of water.

Going back first to the variables (τ, ξ), then to the dimensionless variables (t^*, x^*), and eventually to the dimensionful variables (t, x) and field δh, the soliton solution reads

[ao] D. Korteweg, 1848–1941 [ap] G. de Vries, 1866–1934

$$\delta h(t,x) = \frac{\delta h_{\max}}{\cosh^2\left\{\frac{1}{2h_0}\sqrt{\frac{3\delta h_{\max}}{h_0}}\left[x - \sqrt{gh_0}\left(1+\frac{\delta h_{\max}}{2h_0}\right)t\right]\right\}}, \tag{VI.62b}$$

with $\delta h_{\max}$ the maximum amplitude of the solitary wave. This solution, represented in Fig. VI.2, has a few properties that can be read directly off its expression and differ from those of linear sea surface waves, namely

- the propagation velocity c_{soliton} of the soliton—which is the factor in front of t—is larger than for linear waves;
- the velocity c_{soliton} increases with the amplitude $\delta h_{\max}$ of the soliton;
- the width of the soliton decreases with its amplitude.

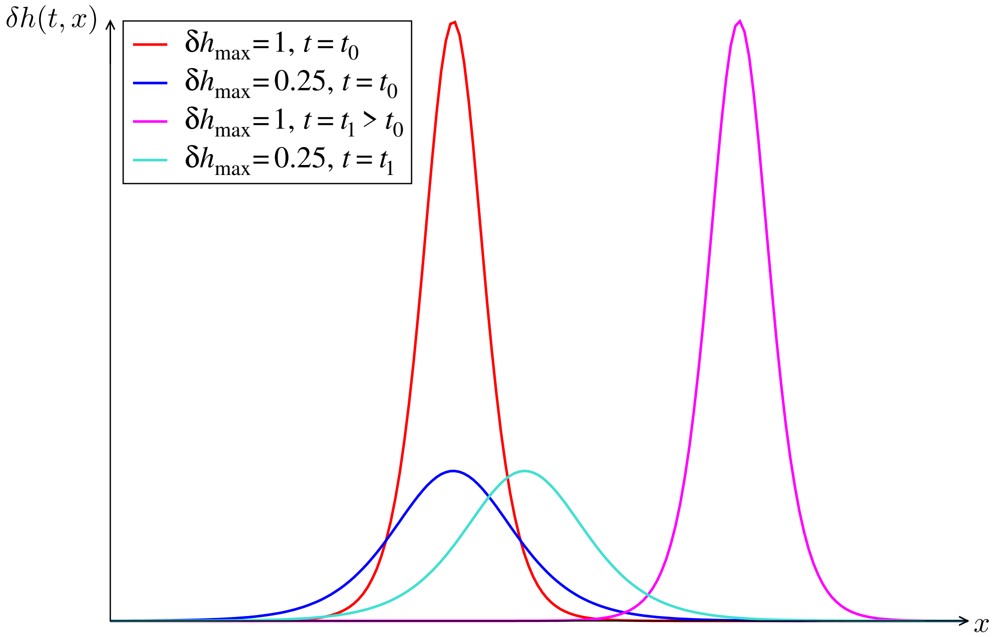

Figure VI.2 – Profile of the soliton solution (VI.62).

Bibliography for Chapter VI

- National Committee for Fluid Mechanics film & film notes on *Waves in Fluids*;
- Guyon *et al.* [2] Chapter 6.4;
- Landau–Lifshitz [4, 5] Chapters I § 12, VIII § 64–65, IX § 84–85, and X § 99;
- Sommerfeld [7, 8] Chapters III § 13, V § 23, 24 & 26 and VII § 37.