V.1 Statics and steady laminar flows of a Newtonian fluid

In this Section, we first write down the equations governing the statics of a Newtonian fluid (§ V.1.1), then we investigate a few idealized stationary laminar fluid motions, in which the velocity field is entirely driven by the no-slip condition at boundaries (Secs. V.1.2–V.1.4).

V.1.1 Static Newtonian fluid

Consider a motionless $[\vec{v}(t,\vec{r}) = \vec{0}]$ Newtonian fluid in an external gravitational potential $\Phi(\vec{r})$—or more generally, submitted to conservative volume forces such that

$$\vec{f}_V(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}). \qquad \text{(IV.1)}$$

The three coupled equations (III.9), (III.31) and (III.36) respectively simplify to

$$\frac{\partial\rho(t,\vec{r})}{\partial t} = 0, \qquad \text{(V.1a)}$$

which means that the mass density $\rho(t,\vec{r})$ is time independent,

$$\vec{\nabla}\mathcal{P}(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}), \qquad \text{(V.1b)}$$

identical to the fundamental equation (IV.2) governing the hydrostatics of a perfect fluid, and

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla}\cdot\left[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\right], \qquad \text{(V.1c)}$$

which describes the transport of energy without macroscopic fluid motion, i.e. non-convectively, thanks to *heat conduction*.

Given an equation of state relating the internal energy density to the temperature, Eq. (V.1c) can become an equation for $T(t,\vec{r})$ only, in particular if the various thermodynamic and transport coefficients involved are assumed to be uniform across the fluid.

V.1.2 Plane Couette flow

In the example of this Section and the next two ones (Secs. V.1.3–V.1.4), we consider steady, incompressible, laminar flows, in absence of significant volume forces. Since the mass density ρ is fixed, thus known, only four equations are needed to determine the flow velocity $\vec{v}(\vec{r})$ and pressure $\mathcal{P}(\vec{r})$, the simplest possibility being to use the continuity and Navier–Stokes equations. In the stationary and incompressible regime, these become

$$\vec{\nabla}\cdot\vec{v}(\vec{r}) = 0 \qquad \text{(V.2a)}$$

$$\left[\vec{v}(\vec{r})\cdot\vec{\nabla}\right]\vec{v}(\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(\vec{r}) + \nu\triangle\vec{v}(\vec{r}), \qquad \text{(V.2b)}$$

with ν the kinematic shear viscosity, assumed to be the same throughout the fluid.

The so-called (plane) *Couette flow*[(ab)] is, in its idealized version, the motion of a viscous fluid between two infinitely extended plane plates, as represented in Fig. V.1, where the lower plate is at rest, while the upper one moves in its own plane with a constant velocity $\vec{u}$. It will be assumed

[(ab)] M. Couette, 1858–1943

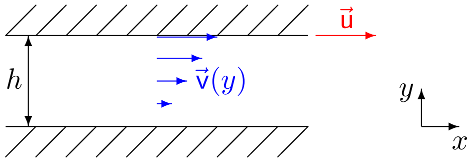

Figure V.1 – Setup of the plane Couette flow.

that the same pressure $\mathcal{P}_\infty$ holds "at infinity" in any direction in the (x,z)-plane.

As the flow is assumed to be laminar, the geometry of the problem is invariant under arbitrary translations in the (x,z)-plane. This is automatically taken into account by the ansatz $\vec{v}(\vec{r}) = v(y)\,\vec{e}_x$ for the flow velocity. Inserting this form in Eqs. (V.2) yields

$$\frac{\partial v(y)}{\partial x} = 0, \tag{V.3a}$$

$$v(y)\frac{\partial v(y)}{\partial x}\,\vec{e}_x = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(\vec{r}) + \nu\frac{d^2v(y)}{dy^2}\,\vec{e}_x. \tag{V.3b}$$

With the ansatz for $\vec{v}(\vec{r})$, the first equation is automatically fulfilled, while the term on the left hand side of the second equation vanishes. Projecting the latter on the y and z directions thus yields $\partial\mathcal{P}(\vec{r})/\partial y = 0$—expressing the assumed absence of sizable effects from gravity—and $\partial\mathcal{P}(\vec{r})/\partial z = 0$—since the problem is independent of z. Along the x direction, one finds

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial x} = \eta\frac{d^2v(y)}{dy^2}. \tag{V.4}$$

Since the right member of this equation is independent of x and z, a straightforward integration gives $\mathcal{P}(\vec{r}) = \alpha(y)x + \beta(y)$, where the functions α, β only depend on y. These functions are determined by the boundary conditions: from $\mathcal{P}(x\!=\!-\infty) = \mathcal{P}(x\!=\!\infty) = \mathcal{P}_\infty$ follow $\alpha(y) = 0$, $\beta(y) = \mathcal{P}_\infty$, and Eq. (V.4) eventually simplifies to

$$\frac{d^2v(y)}{dy^2} = 0.$$

This yields $v(y) = \gamma y + \delta$, with two integration constants γ and δ that are again fixed by the boundary conditions. At each plate, the relative velocity of the fluid with respect to the plate must vanish:

$$v(y\!=\!0) = 0, \qquad v(y\!=\!h) = |\vec{u}|,$$

leading to $\delta = 0$ and $\gamma = |\vec{u}|/h$. All in all, the velocity thus depends linearly on y

$$\vec{v}(\vec{r}) = \frac{y}{h}\,\vec{u} \quad \text{for } 0 \leq y \leq h.$$

Consider now a surface element $d^2\mathcal{S}$. The contact force $d^2\vec{F}_s$ exerted on it by the fluid follows from the Cauchy stress tensor, whose Cartesian components (III.27c) here read

$$\sigma^{ij}(\vec{r}) = -\mathcal{P}(\vec{r})\delta^{ij} + \eta\left[\frac{\partial v^i(\vec{r})}{\partial x_j} + \frac{\partial v^j(\vec{r})}{\partial x_i}\right] \cong \begin{pmatrix} -\mathcal{P}_\infty & \eta\frac{|\vec{u}|}{h} & 0 \\ \eta\frac{|\vec{u}|}{h} & -\mathcal{P}_\infty & 0 \\ 0 & 0 & -\mathcal{P}_\infty \end{pmatrix}.$$

The force per unit surface on the motionless plate at $y = 0$, corresponding to a unit normal vector $\vec{e}_n(\vec{r}) = \vec{e}_y$, is

$$\frac{d^2\vec{F}_s(\vec{r})}{d^2\mathcal{S}} = \vec{\mathcal{T}}_s(\vec{r}) = \left[\sum_{i,j=1}^{3}\sigma^{ij}(\vec{r})\,\vec{e}_i\otimes\vec{e}_j\right]\cdot\vec{e}_y = \sum_{i,j=1}^{3}\sigma^{ij}(\vec{r})\big(\vec{e}_j\cdot\vec{e}_y\big)\vec{e}_i = \begin{pmatrix} \eta\frac{|\vec{u}|}{h} \\ -\mathcal{P}_\infty \\ 0 \end{pmatrix}.$$

Due to the friction exerted by the fluid, the lower plate is dragged by the flow in the (positive) x direction.

Remark: The tangential stress on the lower plate is $\eta \vec{\mathsf{u}}/h$, proportional to the shear viscosity: measuring the tangential stress with known $|\vec{\mathsf{u}}|$ and h yields a measurement of η. In practice, this measurement rather involves the more realistic cylindrical analog to the above plane flow, the so-called *Couette–Taylor flow*.[(ac)]

V.1.3 Plane Poiseuille flow

Let us now consider the flow of a Newtonian fluid between two motionless plane plates with a finite length along the x direction—yet still infinitely extended along the z direction—, as illustrated in Fig. V.2. The pressure is assumed to be different at both ends of the plates in the x direction, leading to the presence of a pressure gradient along x.

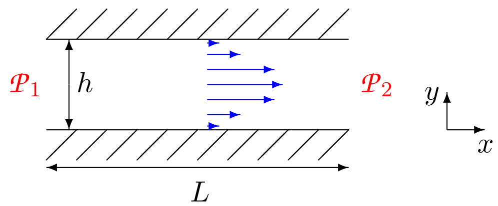

Figure V.2 – Flow between two motionless plates for $\mathcal{P}_1 > \mathcal{P}_2$, i.e. $\delta\mathcal{P} > 0$.

Assuming for the flow velocity $\vec{\mathsf{v}}(\vec{r})$ the same form $\mathsf{v}(y)\,\vec{e}_x$, independent of x, as in the case of the plane Couette flow, the equations of motion governing $\mathsf{v}(y)$ and pressure $\mathcal{P}(\vec{r})$ are the same as in the previous § V.1.2, namely Eqs. (V.3)–(V.4). The boundary conditions are however different. Thus, $\mathcal{P}_1 \neq \mathcal{P}_2$ results in a finite constant pressure gradient along x, $\alpha = \partial\mathcal{P}(\vec{r})/\partial x = -\delta\mathcal{P}/L \neq 0$, with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$ the pressure drop. Equation (V.4) then leads to

$$\mathsf{v}(y) = -\frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}y^2 + \gamma y + \delta,$$

with γ and δ two new constants.

The "no-slip" boundary conditions for the velocity at the two plates read

$$\mathsf{v}(y\!=\!0) = 0, \qquad \mathsf{v}(y\!=\!h) = 0,$$

which leads to $\delta = 0$ and $\gamma = \dfrac{1}{2\eta}\dfrac{\delta\mathcal{P}}{L}h$. The flow velocity thus has the parabolic profile

$$\mathsf{v}(y) = \frac{1}{2\eta}\frac{\delta\mathcal{P}}{L}\big[y(h-y)\big] \quad \text{for } 0 \leq y \leq h, \tag{V.5}$$

directed along the direction of the pressure gradient.

Remark: The flow velocity (V.5) becomes clearly problematic in the limit $\eta \to 0$! Tracing the problem back to its source, the equations of motion (V.3) cannot hold with a finite pressure gradient along the x direction and a vanishing viscosity. One quickly checks that the only possibility in the case of a perfect fluid is to drop one of the assumptions, either incompressibility or laminarity.

V.1.4 Hagen–Poiseuille flow

The previous two examples involved plates with an infinite length in at least one direction, thus were idealized constructions. In contrast, an experimentally realizable fluid motion is that of the *Hagen–Poiseuille flow*,[(ad)] in which a Newtonian fluid flows under the influence of a pressure

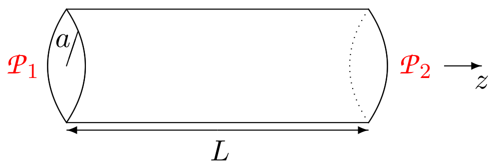

Figure V.3 – Setup of the Hagen–Poiseuille flow.

gradient in a cylindrical tube with finite length L and radius a (Fig. V.3). Again, the motion is assumed to be steady, incompressible and laminar.

Using cylindrical coordinates, the ansatz $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}(r)\,\vec{e}_z$ with $r = \sqrt{x^2+y^2}$ satisfies the continuity equation $\vec{\nabla}\cdot\vec{\mathsf{v}}(\vec{r}) = 0$ and gives for the incompressible Navier–Stokes equation

$$\vec{\nabla}\mathcal{P}(\vec{r}) = \eta\triangle\vec{\mathsf{v}}(\vec{r}) \quad\Leftrightarrow\quad \begin{cases} \dfrac{\partial\mathcal{P}(\vec{r})}{\partial x} = \dfrac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0 \\ \dfrac{\partial\mathcal{P}(\vec{r})}{\partial z} = \eta\left[\dfrac{\partial^2\mathsf{v}(r)}{\partial x^2} + \dfrac{\partial^2\mathsf{v}(r))}{\partial y^2}\right] = \eta\left[\dfrac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}\mathsf{v}(r)}{\mathrm{d}r}\right]. \end{cases} \tag{V.6}$$

The right member of the equation in the second line is independent of z, implying that the pressure gradient along the z direction is constant. Using the boundary conditions yields

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\frac{\delta\mathcal{P}}{L},$$

with $\delta\mathcal{P} \equiv \mathcal{P}_1 - \mathcal{P}_2$. The z component of the Navier–Stokes equation (V.6) thus becomes

$$\frac{\mathrm{d}^2\mathsf{v}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\mathsf{v}}{\mathrm{d}r} = -\frac{\delta\mathcal{P}}{\eta L}. \tag{V.7}$$

As always, this linear differential equation is solved in two successive steps, starting with the associated homogeneous equation. To find the general solution of the latter, one may introduce $\chi(r) \equiv \mathrm{d}\mathsf{v}(r)/\mathrm{d}r$, which obeys the simpler equation

$$\frac{\mathrm{d}\chi(r)}{\mathrm{d}r} + \frac{\chi(r)}{r} = 0.$$

The generic solution is $\ln\chi(r) = -\ln r + \text{const.}$, i.e. $\chi(r) = A/r$ with A a constant. This then leads to $\mathsf{v}(r) = A\ln r + B$ with B an additional constant.

A particular solution of the inhomogeneous equation (V.7) is $\mathsf{v}(r) = Cr^2$ with $C = -\delta\mathcal{P}/4\eta L$. The general solution of Eq. (V.7) is then given by

$$\mathsf{v}(r) = A\ln r + B - \frac{\delta\mathcal{P}}{4\eta L}r^2,$$

where the two integration constants still need to be determined.

To have a regular flow velocity at $r = 0$, the constant A should vanish. In turn, the boundary condition at the tube wall, $\mathsf{v}(r\!=\!a) = 0$, determines the value of the constant $B = (\delta\mathcal{P}/4\eta L)a^2$. All in all, the velocity profile thus reads

$$\mathsf{v}(r) = \frac{\delta\mathcal{P}}{4\eta L}\big(a^2 - r^2\big) \quad \text{for } r \leq a. \tag{V.8}$$

This is again parabolic, with $\vec{\mathsf{v}}$ pointing in the same direction as the pressure drop.

The mass flow rate across the tube cross section follows from a straightforward integration:

$$Q = \int_0^a \rho\mathsf{v}(r)\,2\pi r\,\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\int_0^a \big(a^2 r - r^3\big)\,\mathrm{d}r = 2\pi\rho\frac{\delta\mathcal{P}}{4\eta L}\frac{a^4}{4} = \frac{\pi\rho a^4}{8\eta}\frac{\delta\mathcal{P}}{L}. \tag{V.9}$$

This result, known as *Hagen–Poiseuille law* (or equation), shows that the mass flow rate is proportional to the pressure drop per unit length.

[ac] G. I. Taylor, 1886–1975 [ad] G. Hagen, 1797–1884

Remarks:

∗ The Hagen–Poiseuille law only holds under the assumption that the flow velocity vanishes at the tube walls. The experimental confirmation of the law—which was actually deduced from experiment by Hagen (1839) and Poiseuille (1840)—is thus a proof of the validity of the no-slip assumption for the boundary condition.

∗ The mass flow rate across the tube cross section may be used to define the average flow velocity such that $Q = \pi a^2 \rho \langle \mathsf{v} \rangle$ with

$$\langle \mathsf{v} \rangle \equiv \frac{1}{\pi a^2} \int_0^a \mathsf{v}(r)\, 2\pi r\, \mathrm{d}r = \frac{1}{2} \mathsf{v}(r=0).$$

The Hagen–Poiseuille law then expresses a proportionality between the pressure drop per unit length and $\langle \mathsf{v} \rangle$ in a laminar flow.

Viewing $\delta\mathcal{P}/L$ as the "generalized force" driving the motion, the corresponding "response" $\langle \mathsf{v} \rangle$ of the fluid is thus linear.

The relation is quite different in the case of a *turbulent* flow with the same geometry: for instance, measurements by Reynolds [21] gave $\delta\mathcal{P}/L \propto \langle \mathsf{v} \rangle^{1.722}$.