

### III.3.3 Newtonian fluid: Navier–Stokes equation

In a real moving fluid, there are *friction forces* that contribute to the transport of momentum between neighboring fluid layers when the latter are in relative motion. Accordingly, the momentum flux-density tensor is no longer given by Eq. (III.21b) or (III.22), but now contains extra terms, involving derivatives of the flow velocity. Accordingly, the Euler equation must be replaced by an alternative dynamical equation, including the friction forces.

#### III.3.3 a Momentum flux density in a Newtonian fluid

The momentum flux density (III.21b) in a perfect fluid only contains two terms—one proportional to the components  $g^{ij}$  of the inverse metric tensor, the other proportional to  $v^i(t, \vec{r})v^j(t, \vec{r})$ . Since the coefficients in front of these two terms could a priori depend on  $\vec{v}^2$ , this represents the most general symmetric tensor of degree 2 which can be constructed with the help of the flow velocity only.

If the use of terms that depend on the spatial derivatives of the velocity field is also allowed, the components of the momentum flux-density tensor can be of the following form, where for the sake of brevity the variables  $t$  and  $\vec{r}$  are omitted

$$\mathbf{T}^{ij} = \mathcal{P}g^{ij} + \rho v^i v^j + A \frac{dv^i}{dx_j} + B \frac{dv^j}{dx_i} + \mathcal{O}\left(\frac{d^2 v_i}{dx_j dx_k}\right) + \dots, \quad (\text{III.25})$$

with coefficients  $A$ ,  $B$  that depend on  $i$ ,  $j$  and on the fluid under consideration.

This ansatz for  $\mathbf{T}^{ij}$ , as well as the form of the energy flux density involved in Eq. (III.36) below, can be “justified” by starting from a microscopic *kinetic theory* of the fluid and writing the solutions of the corresponding equation of motion as a specific expansion—which turns out to be in powers of the Knudsen number (I.4). This also explains why terms of the type  $v^i \partial \mathcal{P} / \partial x_j$  or  $v^i \partial T / \partial x_j$ , with  $T$  the temperature, were not considered in Eq. (III.25).

Despite these theoretical considerations, in the end the actual justification for the choices of momentum or energy flux density is the agreement with the measured behavior of fluids.

As discussed in § I.1.3, the description of a system of particles as a continuous medium, and in particular as a fluid, in local thermodynamic equilibrium, rests on the assumption that the macroscopic quantities of relevance for the medium vary slowly both in space and time. Accordingly, (spatial) gradients should be small: the third and fourth terms in Eq. (III.25) should thus be on the one hand much smaller than the first two ones, on the other hand much larger than the rightmost term as well as those involving higher-order derivatives or of powers of the first derivatives. Neglecting these smaller terms, one obtains “first-order dissipative fluid dynamics”, which describes the motion of *Newtonian fluids*—this actually defines the latter.

Using the necessary symmetry of  $\mathbf{T}^{ij}$ , the third and fourth terms in Eq. (III.25) can be rewritten as the sum of a traceless symmetric contribution and a tensor proportional to the inverse metric tensor. This leads to the momentum flux-density tensor

$$\begin{aligned} \mathbf{T}^{ij}(t, \vec{r}) = & \mathcal{P}(t, \vec{r}) g^{ij}(t, \vec{r}) + \rho(t, \vec{r}) v^i(t, \vec{r}) v^j(t, \vec{r}) \\ & - \eta(t, \vec{r}) \left[ \frac{dv^i(t, \vec{r})}{dx_j} + \frac{dv^j(t, \vec{r})}{dx_i} - \frac{2}{3} g^{ij}(t, \vec{r}) \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] \\ & - \zeta(t, \vec{r}) g^{ij}(t, \vec{r}) \vec{\nabla} \cdot \vec{v}(t, \vec{r}). \end{aligned} \quad (\text{III.26a})$$

In geometric formulation, this reads

$$\mathbf{T}(t, \vec{r}) = \mathcal{P}(t, \vec{r}) \mathbf{g}^{-1}(t, \vec{r}) + \rho(t, \vec{r}) \vec{v}(t, \vec{r}) \otimes \vec{v}(t, \vec{r}) + \boldsymbol{\pi}(t, \vec{r}) \quad (\text{III.26b})$$

where dissipative effects are encoded in the *viscous stress tensor* (xxxvii)

for a Newtonian fluid:

$$\boldsymbol{\pi}(t, \vec{r}) \equiv -2\eta(t, \vec{r}) \left[ \mathbf{D}(t, \vec{r}) - \frac{1}{3} [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}) \right] - \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}) \quad (\text{III.26c})$$

with  $\mathbf{D}(t, \vec{r})$  the strain rate tensor discussed in § II.1.3. Component-wise, this reads

$$\pi^{ij}(t, \vec{r}) \equiv -2\eta(t, \vec{r}) \left[ \mathbf{D}^{ij}(t, \vec{r}) - \frac{1}{3} [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] g^{ij}(t, \vec{r}) \right] - \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] g^{ij}(t, \vec{r}). \quad (\text{III.26d})$$

In terms of the traceless rate-of-shear tensor (II.17b) or of its components (II.17d), one may alternatively write

$$\boldsymbol{\pi}(t, \vec{r}) \equiv -2\eta(t, \vec{r}) \mathbf{S}(t, \vec{r}) - \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}) \quad (\text{III.26e})$$

$$\pi^{ij}(t, \vec{r}) \equiv -2\eta(t, \vec{r}) \mathbf{S}^{ij}(t, \vec{r}) - \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] g^{ij}(t, \vec{r}). \quad (\text{III.26f})$$

This viscous stress tensor involves two novel characteristics of the medium, so-called *transport coefficients*, namely:

- the (dynamical) *shear viscosity* (xxxviii)  $\eta$ , which multiplies the traceless symmetric part of the velocity gradient tensor, i.e. the conveniently termed rate-of-shear tensor;
- the *bulk viscosity*, also called *second viscosity* (xxxix)  $\zeta$ , which multiplies the volume-expansion part of the velocity gradient tensor, i.e. the term proportional to  $\vec{\nabla} \cdot \vec{v}(t, \vec{r})$ .

(xxxvii) viskoser Spannungstensor (xxxviii) Scherviskosität (xxxix) Dehnviskosität, Volumenviskosität, zweite Viskosität

The two corresponding contributions represent a *diffusive* transport of momentum in the fluid—representing a third type of transport besides the convective and thermal ones.

**Remarks:**

\* In the case of a Newtonian fluid, the viscosity coefficients  $\eta$  and  $\zeta$  are independent of the flow velocity. However, they still depend on the temperature and pressure of the fluid, so that they are not necessarily uniform and constant in a real flowing fluid.

\* In an incompressible flow,  $\vec{\nabla} \cdot \vec{v}(t, \vec{r}) = 0$ , the last contribution to the momentum flux density (III.26) drops out. Thus, the bulk viscosity  $\zeta$  only plays a role in compressible fluid motions.<sup>(6)</sup>

\* Expression (III.26c) or (III.26d) of the viscous stress tensor assumes implicitly that the fluid is (locally) isotropic, since the coefficients  $\eta$ ,  $\zeta$  are independent of the directions  $i$ ,  $j$ .

### III.3.3b Surface forces in a Newtonian fluid

The Cauchy stress tensor corresponding to the momentum flux density (III.26) of a Newtonian fluid is

$$\boldsymbol{\sigma}(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) \mathbf{g}^{-1}(t, \vec{r}) - \boldsymbol{\pi}(t, \vec{r}) \quad (\text{III.27a})$$

that is, using the form (III.26e) of the viscous stress tensor

$$\boldsymbol{\sigma}(t, \vec{r}) = -\mathcal{P}(t, \vec{r}) \mathbf{g}^{-1}(t, \vec{r}) + 2\eta(t, \vec{r}) \mathbf{S}(t, \vec{r}) + \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] \mathbf{g}^{-1}(t, \vec{r}). \quad (\text{III.27b})$$

Component-wise, this becomes

$$\sigma^{ij}(t, \vec{r}) = \left\{ -\mathcal{P}(t, \vec{r}) + \left[ \zeta(t, \vec{r}) - \frac{2}{3}\eta(t, \vec{r}) \right] \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right\} g^{ij}(t, \vec{r}) + \eta(t, \vec{r}) \left[ \frac{dv^i(t, \vec{r})}{dx_j} + \frac{dv^j(t, \vec{r})}{dx_i} \right]. \quad (\text{III.27c})$$

Accordingly, the mechanical stress vector on an infinitesimally small surface element situated at point  $\vec{r}$  with unit normal vector  $\vec{e}_n(\vec{r})$  reads

$$\begin{aligned} \vec{T}_s(t, \vec{r}) = \boldsymbol{\sigma}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = \sum_{i,j=1}^3 \left\{ \left[ -\mathcal{P}(t, \vec{r}) + \left( \zeta(t, \vec{r}) - \frac{2}{3}\eta(t, \vec{r}) \right) \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right] g^i_j(t, \vec{r}) \right. \\ \left. + \eta(t, \vec{r}) \left( \frac{dv^i(t, \vec{r})}{dx^j} + \frac{dv_j(t, \vec{r})}{dx_i} \right) \right\} n^j(\vec{r}) \vec{e}_i(t, \vec{r}), \quad (\text{III.28}) \end{aligned}$$

with  $n^j(\vec{r})$  the coordinate of  $\vec{e}_n(\vec{r})$  along direction  $j$ . One easily identifies the two components of this stress vector (cf. § I.4.1)

- the term proportional to  $\sum g^i_j n^j \vec{e}_i = \vec{e}_n$  is the *normal stress* on the surface element. It consists of the usual hydrostatic pressure term  $-\mathcal{P} \vec{e}_n$ , and a second contribution, proportional to the local expansion rate  $\vec{\nabla} \cdot \vec{v}$ : in the compressible motion of a Newtonian—and more generally a dissipative—fluid, the normal stress is thus not only given by  $-\mathcal{P} \vec{e}_n$ , but includes additional contributions that vanish in the static case.
- the remaining term is the *tangential stress*, proportional to the shear viscosity  $\eta$ . Accordingly, the value of the latter can be deduced from a measurement of the tangential force acting on a surface element, see § ??.

As in § III.3.2a, the external contact forces acting on a fluid element delimited by a surface  $\mathcal{S}$  can easily be computed. Invoking the Stokes theorem yields

<sup>(6)</sup>As a consequence, the bulk viscosity is often hard to measure—one has to devise a compressible flow—so that it is actually not so well known for many substances, even well-studied ones [13].

$$\begin{aligned}
\oint_S \vec{T}_s(t, \vec{r}) d^2\mathcal{S} &= -\oint_S \mathcal{P}(t, \vec{r}) \vec{e}_n(\vec{r}) d^2\mathcal{S} - \oint_S \boldsymbol{\pi}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) d^2\mathcal{S} \\
&= -\int_{\mathcal{V}} \vec{\nabla} \mathcal{P}(t, \vec{r}) d^3\mathcal{V} + \int_{\mathcal{V}} \vec{\nabla} \cdot \boldsymbol{\pi}(t, \vec{r}) d^3\mathcal{V} \\
&= -\int_{\mathcal{V}} \vec{\nabla} \mathcal{P}(t, \vec{r}) d^3\mathcal{V} + \int_{\mathcal{V}} \vec{f}_{\text{visc}}(t, \vec{r}) d^3\mathcal{V},
\end{aligned} \tag{III.29a}$$

with the local viscous friction force density

$$\begin{aligned}
\vec{f}_{\text{visc}}(t, \vec{r}) &= \sum_{i,j=1}^3 \frac{d}{dx^i} \left\{ \eta(t, \vec{r}) \left[ \frac{dv^i(t, \vec{r})}{dx_j} + \frac{dv^j(t, \vec{r})}{dx_i} \right] \right\} \vec{e}_j(t, \vec{r}) \\
&\quad + \vec{\nabla} \left\{ \left[ \zeta(t, \vec{r}) - \frac{2}{3} \eta(t, \vec{r}) \right] \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right\}.
\end{aligned} \tag{III.29b}$$

### III.3.3c Navier–Stokes equation

Combining the viscous force (III.29b) with the generic equations (III.12), (III.14) and (III.15), the application of Newton's second law to a volume  $\mathcal{V}$  of fluid leads to an identity between sums of volume integrals. Since this relation holds for any volume  $\mathcal{V}$ , it translates into an identity between the integrands, namely

$$\rho(t, \vec{r}) \left\{ \frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} = -\vec{\nabla} \mathcal{P}(t, \vec{r}) + \vec{f}_{\text{visc}}(t, \vec{r}) + \vec{f}_V(t, \vec{r}) \tag{III.30a}$$

or component-wise

$$\begin{aligned}
\rho(t, \vec{r}) \left\{ \frac{\partial v^i(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] v^i(t, \vec{r}) \right\} &= -\frac{d\mathcal{P}(t, \vec{r})}{dx_i} + \frac{d}{dx_i} \left\{ \left[ \zeta(t, \vec{r}) - \frac{2}{3} \eta(t, \vec{r}) \right] \vec{\nabla} \cdot \vec{v}(t, \vec{r}) \right\} \\
&\quad + \sum_{j=1}^3 \frac{d}{dx^j} \left\{ \eta(t, \vec{r}) \left[ \frac{dv^i(t, \vec{r})}{dx_j} + \frac{dv^j(t, \vec{r})}{dx_i} \right] \right\} + [f_V(t, \vec{r})]^i
\end{aligned} \tag{III.30b}$$

for  $i = 1, 2, 3$ .

If the implicit dependence of the viscosity coefficients  $\eta$  and  $\zeta$  on time and position is negligible, one may take them outside of the spatial derivatives. As a result, one obtains the (compressible) Navier<sup>(i)</sup>–Stokes<sup>(j)</sup> equation

$$\rho(t, \vec{r}) \left\{ \frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) \right\} = -\vec{\nabla} \mathcal{P}(t, \vec{r}) + \eta \Delta \vec{v}(t, \vec{r}) + \left( \zeta + \frac{\eta}{3} \right) \vec{\nabla} [\vec{\nabla} \cdot \vec{v}(t, \vec{r})] + \vec{f}_V(t, \vec{r}) \tag{III.31}$$

with  $\Delta = \vec{\nabla}^2$  the Laplacian. This is a non-linear partial differential equation of second order, while the Euler equation (III.18) is of first order.

The difference between the order of the equations is not a mere detail: while the Euler equation looks like the limit  $\eta, \zeta \rightarrow 0$  of the Navier–Stokes equation, this may not necessarily hold true for their solutions. This is for instance due to the fact that their respective boundary conditions differ.

In the case of an incompressible flow, the local expansion rate in the Navier–Stokes equation (III.31) vanishes, leading to the *incompressible Navier–Stokes equation*

$$\frac{\partial \vec{v}(t, \vec{r})}{\partial t} + [\vec{v}(t, \vec{r}) \cdot \vec{\nabla}] \vec{v}(t, \vec{r}) = -\frac{1}{\rho} \vec{\nabla} \mathcal{P}(t, \vec{r}) + \nu \Delta \vec{v}(t, \vec{r}), \tag{III.32}$$

<sup>(i)</sup>C.-L. NAVIER, 1785–1836    <sup>(j)</sup>G. G. STOKES, 1819–1903

with

$$\nu \equiv \eta/\rho \tag{III.33}$$

the *kinematic shear viscosity*.

**Remark:** The dimension of the dynamic viscosity coefficients  $\eta$ ,  $\zeta$  is  $\text{ML}^{-1}\text{T}^{-1}$  and the corresponding unit in the SI system is the Poiseuille<sup>(k)</sup>, abbreviated  $\text{Pa}\cdot\text{s}$ . In contrast, the kinematic viscosity has dimension  $\text{L}^2\text{T}^{-1}$ , i.e. depends only on space and time, hence its denomination.

### III.3.3 d Boundary conditions

At the interface between a viscous fluid, in particular a Newtonian one, and another body—be it an obstacle in the flow, a wall containing the fluid, or even a second viscous fluid which is immiscible with the first one—the relative velocity between the fluid and the body must vanish. This holds not only for the normal component of the velocity (“impermeability” condition), as in perfect fluids, but also for the tangential one, to account for the friction forces. The latter requirement is often referred to as *no-slip condition*.

### III.3.4 Higher-order dissipative fluid dynamics

Instead of considering only the first spatial derivatives of the velocity field in the momentum flux-density tensor (III.25), one may wish to also include the second derivatives, or even higher ones. Such assumptions lead to partial differential equations of motion, replacing the Navier–Stokes equation, of increasing order: *Burnett equation*, *super Burnett equation* [14, 15].

The domain of validity of such higher-order dissipative fluid models is a priori larger than that of first-order fluid dynamics, since it becomes possible to account for stronger gradients. On the other hand, this is at the cost of introducing a large number of new parameters besides the transport coefficients already present in Newtonian fluids. In parallel, the numerical implementation of the model becomes more involved, so that a macroscopic description does not necessarily represent the best approach.

<sup>(k)</sup>J.-L.-M. POISEUILLE, 1797–1869

### III.4.2 Energy conservation in Newtonian fluids

In a real fluid, there exist further mechanisms for transporting energy besides the convective transport due to the fluid motion, namely *diffusion*, either of momentum or of energy:

- On the one hand, the viscous friction forces in the fluid, which already lead to the transport of momentum between neighboring fluid layers moving with different velocities, exert some work in the motion, which induces a diffusive transport of energy. This is accounted for by a contribution  $\boldsymbol{\pi} \cdot \vec{\mathbf{v}}$  to the energy flux density—component-wise, a contribution  $\sum_j \pi_j^i v^j$  to the  $i$ -th component of the flux density—, with  $\boldsymbol{\pi}$  the viscous stress tensor, given in the case of a Newtonian fluid by Eq. (III.26c).
- On the other hand, there is also *heat conduction* from the regions with higher temperatures towards those with lower temperatures. This transport is described by the introduction in the energy flux density of a heat current<sup>(xI)</sup>  $\vec{j}_Q(t, \vec{r}) = -\kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r})$ —in accordance with the local formulation of *Fourier's law*,<sup>(I)</sup> see e.g. Sec. 1.2.1 in Ref. [2]—, with  $\kappa$  the *heat conductivity*<sup>(xII)</sup> of the fluid.

Taking into account these additional contributions, the local formulation of energy conservation in a Newtonian fluid reads

$$\begin{aligned}
 & \frac{\partial}{\partial t} \left[ \frac{1}{2} \rho(t, \vec{r}) \vec{\mathbf{v}}(t, \vec{r})^2 + e(t, \vec{r}) + \rho(t, \vec{r}) \Phi(t, \vec{r}) \right] \\
 & + \vec{\nabla} \cdot \left\{ \left[ \frac{1}{2} \rho(t, \vec{r}) \vec{\mathbf{v}}(t, \vec{r})^2 + e(t, \vec{r}) + \mathcal{P}(t, \vec{r}) + \rho(t, \vec{r}) \Phi(t, \vec{r}) \right] \vec{\mathbf{v}}(t, \vec{r}) \right. \\
 & \quad \left. - \eta(t, \vec{r}) \left[ \left( \vec{\mathbf{v}}(t, \vec{r}) \cdot \vec{\nabla} \right) \vec{\mathbf{v}}(t, \vec{r}) + \vec{\nabla} \left( \frac{\vec{\mathbf{v}}(t, \vec{r})^2}{2} \right) \right] \right. \\
 & \quad \left. - \left[ \zeta(t, \vec{r}) - \frac{2\eta(t, \vec{r})}{3} \right] \left[ \vec{\nabla} \cdot \vec{\mathbf{v}}(t, \vec{r}) \right] \vec{\mathbf{v}}(t, \vec{r}) - \kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r}) \right\} = 0.
 \end{aligned} \tag{III.36}$$

with  $\Phi$  the potential energy per unit mass due to conservative external volume forces. If the three transport coefficients  $\eta$ ,  $\zeta$  and  $\kappa$  vanish, this equation simplifies to that for perfect fluids, Eq. (III.34).

**Remark:** The energy flux density can be read off Eq. (III.36), since it represents the term between curly brackets. Dropping the external potential  $\Phi$ , one can check that it can also be written as

$$\begin{aligned}
 & \left[ \frac{1}{2} \rho(t, \vec{r}) \vec{\mathbf{v}}(t, \vec{r})^2 + e(t, \vec{r}) + \mathcal{P}(t, \vec{r}) \right] \vec{\mathbf{v}}(t, \vec{r}) - 2\eta(t, \vec{r}) \mathbf{S}(t, \vec{r}) \cdot \vec{\mathbf{v}}(t, \vec{r}) \\
 & - \zeta(t, \vec{r}) \left[ \vec{\nabla} \cdot \vec{\mathbf{v}}(t, \vec{r}) \right] \vec{\mathbf{v}}(t, \vec{r}) - \kappa(t, \vec{r}) \vec{\nabla} T(t, \vec{r}), \tag{III.37}
 \end{aligned}$$

<sup>(xI)</sup> Wärmestromvektor    <sup>(xII)</sup> Wärmeleitfähigkeit

<sup>(I)</sup> J. B. FOURIER, 1768–1830

with  $\mathbf{S}(t, \vec{r})$  the traceless symmetric rate-of-shear tensor. One recognizes the various physical sources of energy transport.

### III.4.3 Entropy balance in Newtonian fluids

In a real fluid, with viscous friction forces and heat conductivity, one can expect a priori that the transformation of mechanical energy into heat will lead in general to an increase in entropy, at least as long as one considers a closed and isolated system.

Consider a volume  $\mathcal{V}$  of flowing Newtonian fluid, delimited by a surface  $\mathcal{S}$  at each point  $\vec{r}$  of which the boundary conditions

$$\vec{v}(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = 0 \quad \text{and} \quad \vec{j}_Q(t, \vec{r}) \cdot \vec{e}_n(\vec{r}) = 0$$

hold, where  $\vec{e}_n(\vec{r})$  denotes the unit normal vector to  $\mathcal{S}$  at  $\vec{r}$ . Physically, these boundary conditions mean that neither matter nor heat flows across the surface  $\mathcal{S}$ , so that the system inside  $\mathcal{S}$  is closed and isolated. To completely exclude energy exchanges with the exterior of  $\mathcal{S}$ , it is also assumed that there are no external volume forces acting on the fluid inside volume  $\mathcal{V}$ . We shall investigate the implications of the continuity equation (III.10), the Navier–Stokes equation (III.31), and the energy conservation equation (III.36) for the total entropy  $S$  of the fluid inside  $\mathcal{V}$ . For the sake of brevity, the variables  $t, \vec{r}$  will be omitted in the remainder of this Section.

Starting with the energy conservation equation (III.36), the contribution

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \vec{v}^2 \right) + \vec{\nabla} \cdot \left[ \left( \frac{1}{2} \rho \vec{v}^2 \right) \vec{v} \right]$$

in its first two lines can be replaced by

$$\rho \vec{v} \cdot \frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \frac{\partial \rho}{\partial t} \vec{v}^2 + \frac{1}{2} \left[ \vec{\nabla} \cdot (\rho \vec{v}) \right] \vec{v}^2 + \sum_{i=1}^3 \rho v_i (\vec{v} \cdot \vec{\nabla}) v^i = \sum_{i=1}^3 \rho v_i \left[ \frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right], \quad (\text{III.38a})$$

where the continuity equation (III.9) was used.

As recalled in Appendix B the fundamental thermodynamic relation  $U = TS - \mathcal{P}\mathcal{V} + \mu N$  gives on the one hand  $e + \mathcal{P} = Ts + \mu n$ , which leads to

$$\vec{\nabla} \cdot [(e + \mathcal{P}) \vec{v}] = T \vec{\nabla} \cdot (s \vec{v}) + \mu \vec{\nabla} \cdot (n \vec{v}) + \vec{v} \cdot (s \vec{\nabla} T + n \vec{\nabla} \mu) = T \vec{\nabla} \cdot (s \vec{v}) + \mu \vec{\nabla} \cdot (n \vec{v}) + \vec{v} \cdot \vec{\nabla} \mathcal{P}, \quad (\text{III.38b})$$

where the second identity follows from the Gibbs–Duhem relation  $d\mathcal{P} = s dT + n d\mu$ . On the other hand, it leads to  $de = T ds + \mu dn$ , which under consideration of the continuity equation for particle number yields

$$\frac{\partial e}{\partial t} = T \frac{\partial s}{\partial t} + \mu \frac{\partial n}{\partial t} = T \frac{\partial s}{\partial t} - \mu \vec{\nabla} \cdot (n \vec{v}). \quad (\text{III.38c})$$

With the help of relations (III.38a)–(III.38c), the energy conservation equation (III.36) can be rewritten as

$$\begin{aligned} \sum_{i=1}^3 \rho v_i \left[ \frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right] + T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s \vec{v}) + \vec{v} \cdot \vec{\nabla} \mathcal{P} = \\ \sum_{i,j=1}^3 \frac{\partial}{\partial x^j} \left[ \eta \left( \frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) v_i \right] + \sum_{i=1}^3 \frac{\partial}{\partial x_i} [\zeta (\vec{\nabla} \cdot \vec{v}) v_i] + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \end{aligned} \quad (\text{III.38d})$$

Multiplying the  $i$ -th component of Eq. (III.30b) by  $v_i$  gives

$$\rho v_i \left[ \frac{\partial v^i}{\partial t} + (\vec{v} \cdot \vec{\nabla}) v^i \right] + v_i \frac{\partial \mathcal{P}}{\partial x_i} = \sum_{j=1}^3 v_i \frac{\partial}{\partial x^j} \left[ \eta \left( \frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \right] + v_i \frac{\partial}{\partial x^i} (\zeta \vec{\nabla} \cdot \vec{v}).$$

Subtracting this identity, summed over  $i = 1, 2, 3$ , from Eq. (III.38d), yields

$$T \frac{\partial s}{\partial t} + T \vec{\nabla} \cdot (s \vec{v}) = \eta \sum_{i,j=1}^3 \frac{\partial v_i}{\partial x^j} \left( \frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) + \zeta (\vec{\nabla} \cdot \vec{v})^2 + \vec{\nabla} \cdot (\kappa \vec{\nabla} T). \quad (\text{III.39})$$

On the right hand side of this equation, one may use the identity

$$\frac{1}{2} \sum_{i,j=1}^3 \left( \frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left( \frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{v} \right) = \sum_{i,j=1}^3 \left( \frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \frac{\partial v_j}{\partial x^i}, \quad (\text{III.40a})$$

which follows from the fact that both symmetric terms  $\partial v_i / \partial x^j$  and  $\partial v_j / \partial x^i$  from the left member give the same contribution, while the term in  $g_{ij}$  yields a zero contribution, since it multiplies a traceless term.

Additionally, one has

$$\vec{\nabla} \cdot (\kappa \vec{\nabla} T) = T \vec{\nabla} \cdot \left( \frac{\kappa \vec{\nabla} T}{T} \right) + \frac{\kappa}{T} (\vec{\nabla} T)^2. \quad (\text{III.40b})$$

All in all, Eqs. (III.39) and (III.40) lead to

$$\begin{aligned} \frac{\partial s}{\partial t} + \vec{\nabla} \cdot (s \vec{v}) - \vec{\nabla} \cdot \left( \frac{\kappa \vec{\nabla} T}{T} \right) &= \frac{\eta}{2T} \sum_{i,j=1}^3 \left( \frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3} g^{ij} \vec{\nabla} \cdot \vec{v} \right) \left( \frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - \frac{2}{3} g_{ij} \vec{\nabla} \cdot \vec{v} \right) \\ &\quad + \frac{\zeta}{T} (\vec{\nabla} \cdot \vec{v})^2 + \kappa \frac{(\vec{\nabla} T)^2}{T^2}. \end{aligned} \quad (\text{III.41a})$$

This may still be recast in the slightly more compact form

$$\begin{aligned} \frac{\partial s(t, \vec{r})}{\partial t} + \vec{\nabla} \cdot \left[ s(t, \vec{r}) \vec{v}(t, \vec{r}) - \kappa(t, \vec{r}) \frac{\vec{\nabla} T(t, \vec{r})}{T(t, \vec{r})} \right] &= \\ \frac{1}{T(t, \vec{r})} \left\{ 2\eta(t, \vec{r}) \mathbf{S}(t, \vec{r}) : \mathbf{S}(t, \vec{r}) + \zeta(t, \vec{r}) [\vec{\nabla} \cdot \vec{v}(t, \vec{r})]^2 + \kappa(t, \vec{r}) \frac{[\vec{\nabla} T(t, \vec{r})]^2}{T(t, \vec{r})} \right\} \end{aligned} \quad (\text{III.41b})$$

with  $\mathbf{S} : \mathbf{S} \equiv \mathbf{S}_{ij} \mathbf{S}^{ij}$  the scalar obtained by doubly contracting the rate-of-shear tensor with itself.

This equation can then be integrated over the  $\mathcal{V}$  occupied by the fluid:

- When computing the integral of the divergence term on the left hand side with the Stokes theorem, it vanishes thanks to the boundary conditions imposed at the surface  $\mathcal{S}$ ;
- the remaining term in the left member is simply the time derivative  $dS/dt$  of the total entropy of the closed system;
- if all three transport coefficients  $\eta$ ,  $\zeta$ ,  $\kappa$  are *positive*, then it is also the case of the three terms on the right hand side.

One thus finds  $\frac{dS}{dt} \geq 0$ , in agreement with the second law of thermodynamics.

#### Remarks:

\* The previous derivation may be seen as a proof that the transport coefficients must be positive to ensure that the second law of thermodynamics holds.

\* If all three transport coefficients  $\eta$ ,  $\zeta$ ,  $\kappa$  vanish, i.e. in the case of a non-dissipative fluid, Eq. (III.41) simply reduces to the entropy conservation equation in perfect fluids (III.35).