V.1 Sound waves

By definition, the phenomenon which in everyday life is referred to as "sound" consists of *small adiabatic pressure perturbations* around a background flow, where adiabatic actually means that the entropy remains constant. In the presence of such a wave, each point in the fluid undergoes alternative *compression* and *rarefaction* processes. That is, these waves are by construction (parts of) a compressible flow.

We shall first consider sound waves on a uniform perfect fluid at rest (§ V.1.1). **What then? Doppler effect? Rarefaction waves?** Eventually, we discuss how viscous effects in a Newtonian lead to the absorption of sound waves (§ V.1.4).

V.1.1 Sound waves in a uniform fluid at rest

Assuming that there are no external forces, a trivial solution of the dynamical equations of perfect fluids is that with uniform and time independent mass density ρ_0 and pressure $\mathcal{P}_0$, with a vanishing flow velocity $\vec{v}_0 = \vec{0}$. Assuming in addition that the particle number N_0 in the fluid is conserved, its total entropy has a fixed value S_0. These conditions will represent the background flow we consider hereafter.

With the various fields that were just specified, a perturbation (V.1) of this background flow reads

$$\rho(t,\vec{r}) = \rho_0 + \delta\rho(t,\vec{r}), \tag{V.2a}$$

$$\mathcal{P}(t,\vec{r}) = \mathcal{P}_0 + \delta\mathcal{P}(t,\vec{r}), \tag{V.2b}$$

$$\vec{v}(t,\vec{r}) = \vec{0} + \delta\vec{v}(t,\vec{r}). \tag{V.2c}$$

The necessary "smallness" of perturbations means for the mass density and pressure terms

$$|\delta\rho(t,\vec{r})| \ll \rho_0, \quad |\delta\mathcal{P}(t,\vec{r})| \ll \mathcal{P}_0. \tag{V.2d}$$

Regarding the velocity, the background flow does not explicitly specify a reference scale, with which the perturbation should be compared. As we shall see below, the reference scale is actually implicitly contained in the equation(s) of state of the fluid under consideration, and the condition of small perturbation reads

$$|\delta\vec{v}(t,\vec{r})| \ll c_s \tag{V.2e}$$

with c_s the *speed of sound* in the fluid.

Inserting the fields (V.2) in the equations of motion (III.9) and (III.18) and taking into account the uniformity and stationarity of the background flow, one finds

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v}(t,\vec{r}) + \vec{\nabla}\cdot\big[\delta\rho(t,\vec{r})\,\delta\vec{v}(t,\vec{r})\big] = 0, \tag{V.3a}$$

$$\big[\rho_0 + \delta\rho(t,\vec{r})\big]\left\{\frac{\partial\delta\vec{v}(t,\vec{r})}{\partial t} + \big[\delta\vec{v}(t,\vec{r})\cdot\vec{\nabla}\big]\delta\vec{v}(t,\vec{r})\right\} + \vec{\nabla}\delta\mathcal{P}(t,\vec{r}) = \vec{0}. \tag{V.3b}$$

The required smallness of the perturbations will help us simplify these equations, in that we shall only keep the leading-order terms in an expansion in which we consider ρ_0, $\mathcal{P}_0$ as zeroth-order quantities while $\delta\rho(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ and $\delta\vec{v}(t,\vec{r})$ are small quantities of first order. Accordingly, the third term in the continuity equation is presumably much smaller than the other two, and may be left aside in a first approximation. Similarly, the contribution of $\delta\rho(t,\vec{r})$ and the convective term within the curly brackets on the left hand side of Eq. (V.3b) may be dropped. The equations describing the coupled evolutions of $\delta\rho(t,\vec{r})$, $\delta\mathcal{P}(t,\vec{r})$ and $\delta\vec{v}(t,\vec{r})$ are thus *linearized*:

$$\frac{\partial\delta\rho(t,\vec{r})}{\partial t} + \rho_0\vec{\nabla}\cdot\delta\vec{v}(t,\vec{r}) = 0, \tag{V.4a}$$

$$\rho_0\frac{\partial\delta\vec{v}(t,\vec{r})}{\partial t} + \vec{\nabla}\delta\mathcal{P}(t,\vec{r}) = \vec{0}. \tag{V.4b}$$

To obtain a closed system of equations, a further relation between the perturbations is needed. This will be provided by thermodynamics, i.e. by the implicit assumption that the fluid at rest is everywhere in a state in which its pressure $\mathcal{P}$ is function of mass density ρ, (local) entropy S, and (local) particle number N, i.e. that there exists a unique relation $\mathcal{P} = \mathcal{P}(\rho, S, N)$ which is valid at each point in the fluid and at every time. Expanding this relation around the (thermodynamic) point corresponding to the background flow, namely $\mathcal{P}_0 = \mathcal{P}(\rho_0, S_0, N_0)$, one may write

$$\mathcal{P}\big(\rho_0 + \delta\rho, S_0 + \delta S, N_0 + \delta N\big) = \mathcal{P}_0 + \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}\delta\rho + \left(\frac{\partial\mathcal{P}}{\partial S}\right)_{\rho,N}\delta S + \left(\frac{\partial\mathcal{P}}{\partial N}\right)_{S,\rho}\delta N,$$

where the derivatives are to be evaluated at the point (ρ_0, S_0, N_0). Here, we wish to consider isentropic perturbations at constant particle number, i.e. both δS and δN vanish, leaving

$$\delta\mathcal{P} = \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N}\delta\rho.$$

For the partial derivative of the pressure, we introduce the notation

$$c_s^2 \equiv \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N} \tag{V.5}$$

where both sides actually depend on ρ_0, S_0 and N_0, yielding

$$\delta\mathcal{P} = c_s^2\,\delta\rho.$$

This thermodynamic relation holds at each point of the fluid at each instant, so that one can now replace $\vec{\nabla}\delta\mathcal{P}(t,\vec{r})$ by $c_s^2\,\vec{\nabla}\delta\rho(t,\vec{r})$ in Eq. (V.4b):

$$\rho_0\frac{\partial\delta\vec{\mathrm{v}}(t,\vec{r})}{\partial t} + c_s^2\,\vec{\nabla}\delta\rho(t,\vec{r}) = \vec{0}. \tag{V.4c}$$

The equations (V.4a), (V.4c) for the perturbations $\delta\rho(t,\vec{r})$ and $\delta\vec{\mathrm{v}}(t,\vec{r})$ are linear first order partial differential equations. Thanks to the linearity, their solutions form a vector space—at least as long as no initial condition has been specified. One can for instance express the solutions as Fourier transforms, i.e. superpositions of plane waves, characterized by their (angular) frequency ω and their wave vector $\vec{k}$. Accordingly, we test the ansatz

$$\delta\rho(t,\vec{r}) = \widetilde{\delta\rho}(\omega,\vec{k})\,\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\vec{k}\cdot\vec{r}}, \qquad \delta\vec{\mathrm{v}}(t,\vec{r}) = \widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k})\,\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\vec{k}\cdot\vec{r}}, \tag{V.6}$$

with respective amplitudes $\widetilde{\delta\rho}$, $\widetilde{\delta\vec{\mathrm{v}}}$ that a priori depend on ω and $\vec{k}$ and are determined by the initial conditions for the problem. In turn, ω and $\vec{k}$ are not necessarily independent from each other, as we shall indeed find hereafter.

With this ansatz, Eqs. (V.4) become

$$-\mathrm{i}\omega\widetilde{\delta\rho}(\omega,\vec{k}) + \mathrm{i}\rho_0\,\vec{k}\cdot\widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k}) = 0 \tag{V.7a}$$

$$-\mathrm{i}\omega\rho_0\,\widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k}) + \mathrm{i}c_s^2\,\vec{k}\,\widetilde{\delta\rho}(\omega,\vec{k}) = \vec{0}. \tag{V.7b}$$

From the second equation, the amplitude $\widetilde{\delta\vec{\mathrm{v}}}(\omega,\vec{k})$ is proportional to $\vec{k}$; in particular, it lies along the same direction. That is, the inner product $\vec{k}\cdot\widetilde{\delta\vec{\mathrm{v}}}$ simply equals the product of the norms of the two vectors.

Omitting from now on the $(\omega,\vec{k})$-dependence of the amplitudes, the inner product of Eq. (V.7b) with $\vec{k}$—which does not lead to any loss of information—allows one to recast the system as

$$\begin{pmatrix} -\omega & \rho_0 \\ c_s^2\vec{k}^2 & -\omega\rho_0 \end{pmatrix}\begin{pmatrix} \widetilde{\delta\rho} \\ \vec{k}\cdot\widetilde{\delta\vec{\mathrm{v}}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A first, trivial solution to this system is $\widetilde{\delta\rho} = 0$, $\widetilde{\delta\vec{\mathrm{v}}} = \vec{0}$, i.e. the absence of any perturbation. In order for non-trivial solutions to exist, the determinant $(\omega^2 - c_s^2\vec{k}^2)\rho_0$ of the system should vanish.

This leads at once to the *dispersion relation*

$$\omega = \pm c_s |\vec{k}|. \tag{V.8}$$

Denoting by $\vec{e}_{\vec{k}}$ the unit vector in the direction of $\vec{k}$, the perturbations $\delta\rho(t,\vec{r})$ and $\delta\vec{v}(t,\vec{r})$ defined by Eq. (V.6), as well as $\delta\mathcal{P}(t,\vec{r}) = c_s^2\,\delta\rho(t,\vec{r})$, are all functions of $c_s t \pm \vec{r}\cdot\vec{e}_{\vec{k}}$. These are thus *traveling waves*,[li] that propagate with the phase velocity $\omega(\vec{k})/|\vec{k}| = c_s$, which is independent of $\vec{k}$. That is, c_s is the *speed of sound*, and the latter is the same for all frequencies.
For instance, for air at $T = 300$ K, the speed of sound is $c_s = 347\ \mathrm{m}\cdot\mathrm{s}^{-1}$.

Air is a diatomic ideal gas, i.e. it has pressure $\mathcal{P} = Nk_{\mathrm{B}}T/\mathcal{V}$ and internal energy $U = \frac{5}{2}Nk_{\mathrm{B}}T$, giving

$$c_s^2 = \left(\frac{\partial\mathcal{P}}{\partial\rho}\right)_{S,N} = -\frac{\mathcal{V}^2}{mN}\left(\frac{\partial\mathcal{P}}{\partial\mathcal{V}}\right)_{S,N} = -\frac{\mathcal{V}^2}{mN}\left[-\frac{Nk_{\mathrm{B}}T}{\mathcal{V}^2} + \frac{Nk_{\mathrm{B}}}{\mathcal{V}}\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N}\right].$$

The thermodynamic relation $\mathrm{d}U = T\,\mathrm{d}S - \mathcal{P}\,\mathrm{d}\mathcal{V} + \mu\,\mathrm{d}N$ yields at constant entropy and particle number

$$\mathcal{P} = -\left(\frac{\partial U}{\partial\mathcal{V}}\right)_{S,N} = -\frac{5}{2}Nk_{\mathrm{B}}\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N} \quad \text{i.e.} \quad Nk_{\mathrm{B}}\left(\frac{\partial T}{\partial\mathcal{V}}\right)_{S,N} = -\frac{2\mathcal{P}}{5} = -\frac{2}{5}\frac{Nk_{\mathrm{B}}T}{\mathcal{V}}.$$

leading to $c_s^2 = \dfrac{7}{5}\dfrac{k_{\mathrm{B}}T}{m_{\mathrm{air}}}$, with $m_{\mathrm{air}} = 29/\mathcal{N}_{\mathrm{A}}\ \mathrm{g}\cdot\mathrm{mol}^{-1}$. □

Remarks:

∗ Instead of c_s^2, one may use the fluid *isentropic compressibility*, defined as

$$\beta_S \equiv -\frac{1}{\mathcal{V}}\left(\frac{\partial\mathcal{V}}{\partial\mathcal{P}}\right)_S = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial\mathcal{P}}\right)_S, \tag{V.9a}$$

to relate the variations of pressure and mass density $\delta\mathcal{P}$, $\delta\rho$. This compressibility is related to c_s (evaluated at $\rho = \rho_0$) by

$$\beta_S = \frac{1}{\rho_0 c_s^2} \quad \text{resp.} \quad c_s = \frac{1}{\sqrt{\rho_0\beta_S}}, \tag{V.9b}$$

which shows that the sound velocity is larger in "stiffer" fluids, i.e. fluids with a smaller compressibility — as generally liquids with respect to gases.

∗ Taking the real parts of the complex quantities in the harmonic waves (V.6), so as to obtain real-valued $\delta\rho$, $\delta\mathcal{P}$ and $\delta\vec{v}$, one sees that these will be alternatively positive and negative, and in average—over a duration much longer than a period $2\pi/\omega$—zero. This in particular means that the successive compression and condensation ($\delta\mathcal{P} > 0$, $\delta\rho > 0$) or depression and rarefaction[lii] ($\delta\mathcal{P} < 0$, $\delta\rho < 0$) processes do not lead to a resulting transport of matter.

∗ A single harmonic wave (V.6) is a traveling wave. Yet if the governing equation or systems of equations is linear or has been linearized, as was done here, the superposition of harmonic waves is a valid solution. In particular, the superposition of two harmonic traveling waves with equal frequencies ω, opposite wave vectors $\vec{k}$—which is allowed by the dispersion relation (V.8)—and equal amplitudes leads to a *standing wave*, in which the dependence on time and space is proportional to $\mathrm{e}^{\mathrm{i}\omega t}\cos(\vec{k}\cdot\vec{r})$.

Coming back to Eq. (V.7b), the proportionality of $\widetilde{\delta\vec{v}}(\omega,\vec{k})$ and $\vec{k}$ means that the sound waves in a fluid are *longitudinal*—in contrast to electromagnetic waves in vacuum, which are transversal waves.

The nonexistence of transversal waves in fluids reflects the absence of forces that would act against shear deformations so as to restore some equilibrium shape—shear viscous effects cannot play that role.

[li] *fortschreitende Wellen* [lii] *Verdünnung*

In contrast, there can be transversal sound waves in elastic solids, as e.g. the so-called S-modes (shear modes) in geophysics.

The inner product of Eq. (V.7b) with $\vec{k}$, together with the dispersion relation (V.8) and the collinearity of $\widetilde{\delta\vec{v}}$ and $\vec{k}$, leads to the relation

$$\omega\rho_0|\vec{k}||\widetilde{\delta\vec{v}}| = c_s^2\,|\vec{k}|\widetilde{\delta\rho} \qquad \Leftrightarrow \qquad \frac{|\widetilde{\delta\vec{v}}|}{c_s} = \frac{\widetilde{\delta\rho}}{\rho_0}$$

for the amplitudes of the perturbations. This justifies condition (V.2e), which is then consistent with (V.2d). Similarly, inserting the ansatz (V.6) in Eq. (V.3b), the terms within curly brackets become $-\mathrm{i}\omega\,\widetilde{\delta\vec{v}} + \mathrm{i}\big(\vec{k}\cdot\widetilde{\delta\vec{v}}\big)\widetilde{\delta\vec{v}}$: again, neglecting the second with respect to the first is equivalent to requesting $\big|\widetilde{\delta\vec{v}}\big| \ll c_s$.

Remark: Going back to Eqs. (V.4a) and (V.4c), the difference of the time derivative of the first one and the divergence of the second one leads to the known *wave equation*[(26)]

$$\frac{\partial^2\delta\rho(t,\vec{r})}{\partial t^2} - c_s^2\triangle\delta\rho(t,\vec{r}) = 0, \tag{V.10a}$$

If the flow—including the background flow on which the sound wave develops, in case $\vec{v}_0$ is not trivial as it was assumed here—is irrotational, so that one may write $\vec{v}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r})$, then the velocity potential φ also obeys the same equation

$$\frac{\partial^2\varphi(t,\vec{r})}{\partial t^2} - c_s^2\triangle\varphi(t,\vec{r}) = 0.$$

V.1.2 Sound waves in a moving fluid

Doppler effect!

V.1.3 Riemann problem. Rarefaction waves

Should be added at some point

(26) This traditional denomination is totally out of place in a chapter in which there are several types of waves, each of which has its own governing "wave equation". Yet historically, due to its role for electromagnetic or sound waves, it is *the* archetypal wave equation, while the equations governing other types of waves often have a specific name.