IV.4 Potential flows

According to Lagrange's theorem (IV.20), every flow of a perfect barotropic fluid with conservative volume forces which is everywhere irrotational at a given instant remains irrotational at every time.

Focusing accordingly on the *incompressible* and irrotational motion of an ideal fluid with conservative volume forces, which is also referred to as a *potential flow*[(xlvii)], the dynamical equations can be recast such that the main one is a linear partial differential equation for the *velocity potential* (§ IV.4.1), for which there exist mathematical results (§ IV.4.2). Two-dimensional potential flows are especially interesting, since one may then introduce a complex velocity potential—and the corresponding complex velocity—, which is a holomorphic function (§ IV.4.3). This allows one to use the full power of complex analysis so as to devise flows around obstacles with various geometries by combining "elementary" solutions and deforming them.

IV.4.1 Equations of motion in potential flows

Using a known result from vector analysis, a vector field whose curl vanishes everywhere on a simply connected domain of $\mathbb{R}^3$ can be written as the gradient of a scalar field. Thus, in the case of an irrotational flow $\vec{\nabla} \times \vec{v}(t,\vec{r}) = \vec{0}$, the velocity field can be expressed as

$$\vec{v}(t,\vec{r}) = -\vec{\nabla}\varphi(t,\vec{r}) \tag{IV.29}$$

with $\varphi(t,\vec{r})$ the so-called *velocity potential*.[(xlviii)]

Remarks:

∗ The minus sign in definition (IV.29) is purely conventional. While the choice adopted here is not universal, it has the advantage of being directly analogous to the convention in electrostatics ($\vec{E} = -\vec{\nabla}\Phi_{\rm Coul.}$) or Newtonian gravitation physics ($\vec{g} = -\vec{\nabla}\Phi_{\rm Newt.}$).

∗ Since Lagrange's theorem does not hold in a dissipative fluid, in which vorticity can be created or annihilated (Sec. ??), the rationale behind the definition of the velocity potential disappears.

Using the velocity potential (IV.29) and the relation $\vec{a}_V = -\vec{\nabla}\Phi$ expressing that the volume forces are conservative, the Euler equation (III.20) reads

$$-\frac{\partial\vec{\nabla}\varphi(t,\vec{r})}{\partial t} + \vec{\nabla}\left\{\frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2} + \Phi(t,\vec{r})\right\} = -\frac{1}{\rho(t,\vec{r})}\vec{\nabla}\mathcal{P}(t,\vec{r}).$$

Assuming that the flow is also incompressible, and thus ρ constant, this becomes

$$-\frac{\partial\vec{\nabla}\varphi(t,\vec{r})}{\partial t} + \vec{\nabla}\left\{\frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2} + \frac{\mathcal{P}(t,\vec{r})}{\rho} + \Phi(t,\vec{r})\right\} = \vec{0}. \tag{IV.30}$$

or equivalently

$$-\frac{\partial\varphi(t,\vec{r})}{\partial t} + \frac{\left[\vec{\nabla}\varphi(t,\vec{r})\right]^2}{2} + \frac{\mathcal{P}(t,\vec{r})}{\rho} + \Phi(t,\vec{r}) = C(t), \tag{IV.31}$$

where $C(t)$ denotes a function of time only.

Eventually, expressing the incompressibility condition [cf. Eq. (II.15)] $\vec{\nabla}\cdot\vec{v}(t,\vec{r}) = 0$ leads to the *Laplace equation*[(w)] for the velocity potential φ

$$\triangle\varphi(t,\vec{r}) = 0. \tag{IV.32}$$

(xlvii) *Potentialströmung* (xlviii) *Geschwindigkeitspotential*

(w) P.-S. (de) Laplace, 1749–1827

Equations (IV.29), (IV.31) and (IV.32) are the three equations of motion governing potential flows. Since the Laplace equation is partial differential, it is still necessary to specify the corresponding boundary conditions.

In agreement with the discussion in § III.3.2 c, there are two types of conditions: at walls or obstacles, which are impermeable to the fluid; and "at infinity"—for a flow in an unbounded domain of space—, where the fluid flow is generally assumed to be uniform. Choosing a proper reference frame $\mathcal{R}$, this uniform motion of the fluid may be turned into having a fluid at rest. Denoting by $\mathcal{S}(t)$ the material surface associated with walls or obstacles, which are assumed to be moving with velocity $\vec{v}_{\rm obs.}$ with respect to $\mathcal{R}$, and by $\vec{e}_n(t,\vec{r})$ the unit normal vector to $\mathcal{S}(t)$ at a given point $\vec{r}$, the condition of vanishing relative normal velocity reads

$$-\vec{e}_n(t,\vec{r})\cdot\vec{\nabla}\varphi(t,\vec{r}) = \vec{e}_n(t,\vec{r})\cdot\vec{v}_{\rm obs.}(t,\vec{r}) \quad \text{on} \quad \mathcal{S}(t). \tag{IV.33a}$$

In turn, the condition of rest at infinity reads

$$\varphi(t,\vec{r}) \underset{|\vec{r}|\to\infty}{\sim} K(t), \tag{IV.33b}$$

where in practice the scalar function $K(t)$ will be given.

Remarks:

∗ Since the Laplace equation (IV.32) is linear—the non-linearity of the Euler equation is in Eq. (IV.31), which becomes trivial once the spatial dependence of the velocity potential has been determined—, it will be possible to *superpose* the solutions of "simple" problems to obtain the solution for a more complicated geometry.

∗ In potential flows, the dependences on time and space are somewhat separated: The Laplace equation (IV.32) governs the spatial dependence of φ and thus $\vec{\mathrm{v}}$; meanwhile, time enters the boundary conditions (IV.33), and is thus used to fix the amplitude of the solution of the Laplace equation. In turn, when φ is known, relation (IV.31) gives the pressure field, where the integration "constant" $C(t)$ will also be fixed by boundary conditions.

IV.4.2 Mathematical results on potential flows

The *boundary value problem* consisting of the Laplace differential equation (IV.32) together with the boundary conditions on normal derivatives (IV.33) is called a *Neumann problem*[x] or boundary value problem of the second kind. For such problems, results on the existence and unicity of solutions have been established, which we shall now state without further proof.[21]

IV.4.2 a Potential flows in simply connected regions

The simplest case is that of a potential flow on a simply connected domain $\mathscr{D}$ of space. $\mathscr{D}$ may be unbounded, provided the condition at infinity is that the fluid should be at rest, Eq. (IV.33b).

> *On a simply connected domain, the Neumann problem (IV.32)–(IV.33) for the velocity potential admits a solution $\varphi(t,\vec{r})$, which is unique up to an additive constant. In turn, the flow velocity field $\vec{\mathrm{v}}(t,\vec{r})$ given by relation (IV.29) is unique.* (IV.34)

For a flow on a simply connected region, the relation (IV.29) between the flow velocity and its potential is "easily" invertible: fixing some reference position $\vec{r}_0$ in the domain, one may write

$$\varphi(t,\vec{r}) = \varphi(t,\vec{r}_0) - \int_{\vec{\gamma}} \vec{\mathrm{v}}(t,\vec{r}\,')\cdot \mathrm{d}\vec{\ell}(\vec{r}\,') \tag{IV.35}$$

where the line integral is taken along *any* path $\vec{\gamma}$ on $\mathscr{D}$ connecting the positions $\vec{r}_0$ and $\vec{r}$.

(21) The Laplace differential equation is dealt with in many textbooks, as e.g. in Ref. [18, Chapters 7–9], [19, Chapter 4], or [20, Chapter VII].

(x) C. Neumann, 1832–1925

> That the line integral only depends on the path extremities $\vec{r}_0$, $\vec{r}$, not on the path itself, is clearly equivalent to Stokes' theorem stating that the circulation of velocity along *any* closed contour in the domain $\mathscr{D}$ is zero—it equals the flux of the vorticity, which is everywhere zero, through a surface delimited by the contour and entirely contained in $\mathscr{D}$.

Thus, $\varphi(t,\vec{r})$ is uniquely defined once the value $\varphi(t,\vec{r}_0)$, which is the arbitrary additive constant mentioned above, has been fixed.

This reasoning no longer holds in a multiply connected domain, as we now further discuss.

IV.4.2 b Potential flows in doubly connected regions

As a matter of fact, in a doubly (or a fortiori multiply) connected domain, there are by definition non-contractible closed paths. Consider for instance the domain $\mathscr{D}$ traversed by an infinite cylinder—which is not part of the domain—of Fig. IV.7. The path going from $\vec{r}_0$ to $\vec{r}_2$ along $\vec{\gamma}_{0\to 2}$ and coming back to $\vec{r}_0$ along $\vec{\gamma}\,'_{0\to 2}$[22] cannot be continuously shrunk to a point without leaving $\mathscr{D}$. This opens the possibility that the line integral in relation (IV.35) could depend on the path connecting two points.

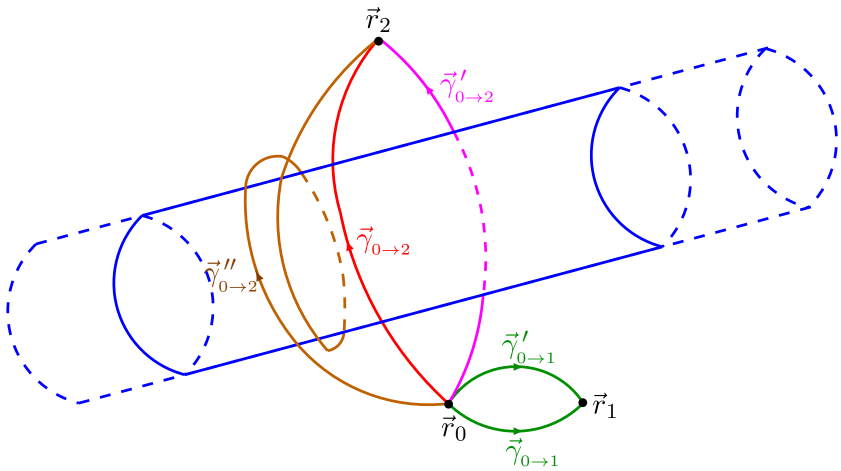

Figure IV.7

In a doubly connected domain $\mathscr{D}$, there is only a single "hole" that prevents closed paths from being homotopic to a point, i.e. contractible. Let $\Gamma(t)$ denote the circulation at time t of the velocity around a closed contour, with a given "positive" orientation, circling the hole *once*. One easily checks—e.g. invoking Stokes' theorem—that this circulation has the same value for all closed paths going only once around the hole with the same orientation, since they can be continuously deformed into each other without leaving $\mathscr{D}$. Accordingly, the "universal" value of the circulation $\Gamma(t)$ is also referred to as *cyclic constant* of the flow.

More generally, the circulation at time t of the velocity around a closed curve circling the hole n times and oriented in the positive resp. negative direction is $n\Gamma(t)$ resp. $-n\Gamma(t)$.

Going back to the line integral in Eq. IV.35, its value will generally depend on the path $\vec{\gamma}$ from $\vec{r}_0$ to $\vec{r}$—or more precisely, on the class, defined by the number of loops around the hole, of the path. Illustrating this idea on Fig. IV.7, while the line integral from $\vec{r}_0$ to $\vec{r}_2$ along the path $\vec{\gamma}_{0\to 2}$ will have a given value $\mathcal{I}$, the line integral along $\vec{\gamma}\,'_{0\to 2}$ will differ by one (say, positive) unit of $\Gamma(t)$ and be equal to $\mathcal{I}+\Gamma(t)$. In turn, the integral along $\vec{\gamma}\,''_{0\to 2}$, which makes one negatively oriented loop more than $\vec{\gamma}_{0\to 2}$ around the cylinder, takes the value $\mathcal{I}-\Gamma(t)$.

These preliminary discussions suggest that if the Neumann problem (IV.32)–(IV.33) for the velocity potential on a doubly connected domain admits a solution $\varphi(t,\vec{r})$, the latter will not be

[22] More precisely, if $\vec{\gamma}\,'_{0\to 2}$ is parameterized by $\lambda \in [0,1]$ when going from $\vec{r}_0$ to $\vec{r}_2$, a path from $\vec{r}_2$ to $\vec{r}_0$ with the same geometric support—which is what is meant by "coming back along $\vec{\gamma}\,'_{0\to 2}$"—is $\lambda \mapsto \vec{\gamma}\,'_{0\to 2}(1-\lambda)$.

a scalar function in the usual sense, but rather a *multivalued* function, whose various values at a given position $\vec{r}$ at a fixed time t differ by an integer factor of the cyclic constant $\Gamma(t)$.

All in all, the following result holds *provided the cyclic constant* $\Gamma(t)$ *is known*, i.e. if its value at time t is part of the boundary conditions:

On a doubly connected domain, the Neumann problem (IV.32)–(IV.33) for the velocity potential with given cyclic constant $\Gamma(t)$ *admits a solution* $\varphi(t,\vec{r})$*, which is unique up to an additive constant. The associated flow velocity field* $\vec{\mathsf{v}}(t,\vec{r})$ *is unique.* (IV.36)

The above wording does not specify the nature of the solution $\varphi(t,\vec{r})$:

- if $\Gamma(t) = 0$, in which case the flow is said to be *acyclic*, the velocity potential $\varphi(t,\vec{r})$ is a univalued function;
- if $\Gamma(t) \neq 0$, i.e. in a *cyclic flow*, the velocity potential $\varphi(t,\vec{r})$ is a multivalued function of its spatial argument. Yet as the difference between the various values at a given $\vec{r}$ is function of time only, the velocity field (IV.29) remains uniquely defined.

Remarks:

∗ Inspecting Eq. (IV.31), one might fear that the pressure field $\mathcal{P}(t,\vec{r})$ could be multivalued, reflecting the term $\partial\varphi(t,\vec{r})/\partial t$. Actually, however, Eq. (IV.31) is a first integral of Eq. (IV.30), in which the $\vec{r}$-independent multiples of $\Gamma(t)$ distinguishing the multiple values of $\varphi(t,\vec{r})$ disappear when the gradient is taken. That is, the term $\partial\varphi(t,\vec{r})/\partial t$ is to be taken with a grain of salt, since in fact it does not contain $\Gamma(t)$ nor its time derivative.

∗ In agreement with the first remark, the reader should remember that the velocity potential $\varphi(t,\vec{r})$ is just a useful auxiliary mathematical function,[23] yet the physical quantity is the velocity itself. Thus the possible multivaluedness of $\varphi(t,\vec{r})$ is not a real physical problem.

IV.4.3 Two-dimensional potential flows

We now focus on two-dimensional potential flows, for which the velocity field—and all other fields—only depend on two coordinates. The latter will either be Cartesian coordinates (x, y), which are naturally combined into a complex variable $z = x + \mathrm{i}y$, or polar coordinates (r, θ). Throughout this Section, the time variable t will not be denoted: apart from possibly influencing the boundary conditions, it plays no direct role in the determination of the velocity potential.

IV.4.3 a Complex flow potential and complex flow velocity

Let us first introduce a few useful auxiliary functions, which either simplify the description of two-dimensional potential flows, or allow one to "generate" such flows at will.

Stream function

Irrespective of whether the motion is irrotational or not, in an incompressible two-dimensional flow one can define a unique (up to an additive constant) *stream function*[xlix] $\psi(x, y)$ such that

$$\mathsf{v}^x(x,y) = -\frac{\partial\psi(x,y)}{\partial y}, \quad \mathsf{v}^y(x,y) = \frac{\partial\psi(x,y)}{\partial x} \tag{IV.37}$$

at every point (x, y). Indeed, when the above two relations hold, the incompressibility criterion $\vec{\nabla}\cdot\vec{\mathsf{v}}(x,y) = 0$ is fulfilled automatically.

[23] Like its cousins: gravitational potential $\Phi_{\text{Newt.}}$, electrostatic potential $\Phi_{\text{Coul.}}$, magnetic vector potential $\vec{A}$…

[xlix] *Stromfunktion*

Remark: As in the case of the relation between the flow velocity field and the corresponding potential, Eq. (IV.29), the overall sign in the relation between $\vec{v}(\vec{r})$ and $\psi(\vec{r})$ is conventional. Yet if one wishes to define the complex flow potential as in Eq. (IV.40) below, the relative sign of $\varphi(\vec{r})$ and $\psi(\vec{r})$ is fixed.

The stream function for a given planar fluid motion is such that the lines along which $\psi(\vec{r})$ is constant are precisely the streamlines of the flow.

Let $\mathrm{d}\vec{x}(\lambda)$ denote a differential line element of a curve $\vec{x}(\lambda)$ of constant $\psi(\vec{r})$, i.e. a curve along which $\vec{\nabla}\psi = \vec{0}$. Then $\mathrm{d}\vec{x}(\lambda)\cdot\vec{\nabla}\psi\big(\vec{x}(\lambda)\big) = 0$ at every point on the line: using relations (IV.37), one recovers Eq. (I.15b) characterizing a streamline. □

Stream functions are also defined in three-dimensional flows, yet in that case two of them are needed. More precisely, one can find two linearly independent functions $\psi_1(\vec{r})$, $\psi_2(\vec{r})$, such that the streamlines are the intersections of the surfaces of constant ψ_1 and of constant ψ_2. That is, they are such that the flow velocity obeys $\vec{v}(\vec{r}) \propto \vec{\nabla}\psi_1(\vec{r}) \times \vec{\nabla}\psi_2(\vec{r})$, with an a priori position-dependent proportionality factor—which can be taken identically equal to unity in an incompressible flow.

Consider now a potential flow, that is a fluid motion which is incompressible and irrotational. In the two-dimensional case, the condition of vanishing vorticity reads

$$\omega^z(x,y) = \frac{\partial \mathsf{v}^y(x,y)}{\partial x} - \frac{\partial \mathsf{v}^x(x,y)}{\partial y} = 0,$$

which under consideration of relations (IV.37) gives

$$\boxed{\triangle\psi(x,y) = 0} \qquad \text{(IV.38a)}$$

at every point (x,y). That is, the stream function obeys the Laplace equation—just like the velocity potential $\varphi(\vec{r})$.

A difference with $\varphi(\vec{r})$ arises with respect to the boundary conditions. At an obstacle or walls, modeled by a "surface" $\mathcal{S}$—in the plane $\mathbb{R}^2$, this surface is rather a curve—, the impermeability condition implies that the velocity is tangential to $\mathcal{S}$, i.e. $\mathcal{S}$ coincides with a streamline:

$$\psi(x,y) = \text{constant on } \mathcal{S} \qquad \text{(IV.38b)}$$

For a flow on an unbounded domain, the velocity is assumed to be uniform at infinity, $\vec{v}(x,y) \to \vec{v}_\infty$, which is the case if

$$\psi(x,y) \underset{|\vec{r}|\to\infty}{\sim} \mathsf{v}_\infty^y\, x - \mathsf{v}_\infty^x\, y \qquad \text{(IV.38c)}$$

with $\mathsf{v}_\infty^x, \mathsf{v}_\infty^y$ the components of $\vec{v}_\infty$.

The boundary conditions (IV.38b)–(IV.38c) on the stream function are thus dissimilar from the corresponding conditions (IV.33a)–(IV.33b) on the velocity potential. In particular, the condition at an obstacle involves the stream function itself, instead of its derivative: the Laplace differential equation (IV.38a) with conditions (IV.38b)–(IV.38c) represents a *Dirichlet problem*,[(y)] or boundary value problem of the first kind, instead of a Neumann problem.

Complex flow potential

In the case of a two-dimensional potential flow, both the velocity potential $\varphi(x,y)$ and the stream function $\psi(x,y)$ are so-called *harmonic functions*, i.e. they are solutions to the Laplace differential equation, see Eqs. (IV.32) and (IV.38a). In addition, gathering Eqs. (IV.29) and (IV.37), one sees that they satisfy at every point (x,y) the identities

$$\frac{\partial\varphi(x,y)}{\partial x} = \frac{\partial\psi(x,y)}{\partial y} \;\big[= -\mathsf{v}^x(x,y)\big], \qquad \frac{\partial\varphi(x,y)}{\partial y} = -\frac{\partial\psi(x,y)}{\partial x} \;\big[= -\mathsf{v}^y(x,y)\big]. \qquad \text{(IV.39)}$$

(y) P. G. (Lejeune-)Dirichlet, 1805–1859

The relations between the partial derivatives of φ and ψ are precisely the Cauchy–Riemann equations obeyed by the corresponding derivatives of the real and imaginary parts of a holomorphic function of a complex variable $z = x + \mathrm{i}y$. That is, the identities (IV.39) suggest the introduction of a *complex (flow) potential*

$$\phi(z) \equiv \varphi(x,y) + \mathrm{i}\psi(x,y) \quad \text{with} \quad z = x + \mathrm{i}y \tag{IV.40}$$

which will automatically be holomorphic on the domain where the flow is defined. The functions φ and ψ are then said to be conjugate to each other. In line with that notion, the curves in the plane along which one of the functions is constant are the field lines of the other, and reciprocally.

Besides the complex potential $\phi(z)$, one also defines the corresponding *complex velocity* as the negative of its derivative, namely

$$\mathsf{w}(z) \equiv -\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} = \mathsf{v}^x(x,y) - \mathrm{i}\mathsf{v}^y(x,y) \tag{IV.41}$$

where the second identity follows at once from the definition of ϕ and the relations between φ or ψ and the flow velocity. Like $\phi(z)$, the complex velocity $\mathsf{w}(z)$ is an analytic function of z.

IV.4.3 b Elementary two-dimensional potential flows

As a converse to the above construction of the complex potential, the real and imaginary parts of *any* analytic function of a complex variable are harmonic functions, i.e. any analytical function $\phi(z)$ defines a two-dimensional potential flow on its domain of definition. Accordingly, we now investigate a few "basic" complex potentials and the flows they describe.

Uniform flow

The simplest possibility is that of a linear complex potential:

$$\phi(z) = -\mathsf{v}\,\mathrm{e}^{-\mathrm{i}\alpha}z \quad \text{with} \quad \mathsf{v} \in \mathbb{R},\ \alpha \in \mathbb{R}. \tag{IV.42}$$

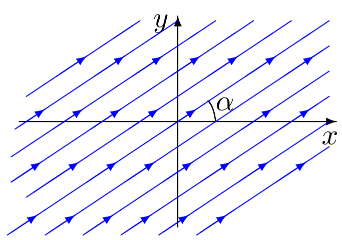

Figure IV.8

Using for instance Eq. (IV.41), this trivially leads to a uniform velocity field making an angle α with the x-direction,

$$\vec{\mathsf{v}}(x,y) = \big(\cos\alpha\,\vec{\mathrm{e}}_x + \sin\alpha\,\vec{\mathrm{e}}_y\big)\mathsf{v},$$

as illustrated in Fig. IV.8, in which a few streamlines are displayed, to which the equipotential lines (not shown) of $\varphi(x,y)$ are perpendicular.

Flow source or sink

Another flow with "simple" streamlines is that defined by the complex potential[24]

$$\phi(z) = -\frac{Q}{2\pi}\log(z - z_0) \quad \text{with} \quad Q \in \mathbb{R},\ z_0 \in \mathbb{C}. \tag{IV.43a}$$

The resulting complex flow velocity

$$\mathsf{w}(z) = \frac{Q}{2\pi(z - z_0)} \tag{IV.43b}$$

has a simple pole at $z = z_0$. Using polar coordinates (r, θ) centered on that pole, the flow velocity

[24] The reader unwilling to take the logarithm of a dimensionful quantity—which is a healthy reaction—may divide $z - z_0$ resp. r by a length in the potentials (IV.43a) and (IV.44a) resp. (IV.43d) and (IV.44c), or write the difference in Eq. (IV.46) as the logarithm of a quotient. She will however quickly convince herself that this does not affect the velocities (IV.43b) and (IV.44b).

is purely radial:

$$\vec{v}(r,\theta) = \frac{Q}{2\pi r}\,\vec{e}_r \tag{IV.43c}$$

as displayed in the left panel of Fig. IV.9 while the flow potential and the stream function are

$$\varphi(r,\theta) = -\frac{Q}{2\pi}\log r, \quad \psi(r,\theta) = -\frac{Q}{2\pi}\,\theta. \tag{IV.43d}$$

By computing the flux of velocity through a closed curve circling the pole—e.g. a circle centered on z_0, which is an equipotential of φ—, one finds that Q represents the mass flow rate through that curve. If Q is positive, there is a *source* of flow at z_0; if Q is negative, the pole at z_0 models a *sink* into which the fluid disappears.

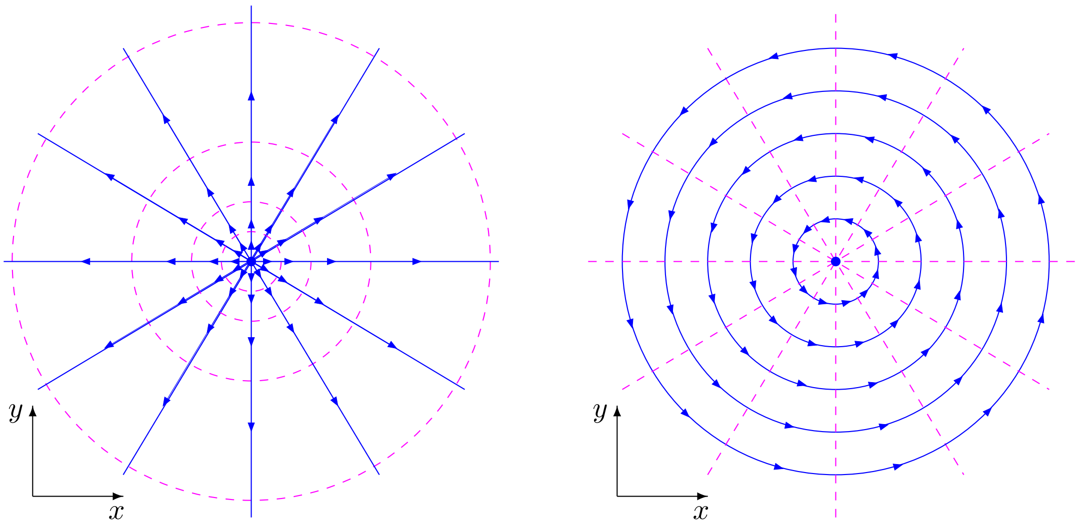

Figure IV.9 – Streamlines (full) and equipotential lines (dashed) for a flow source (IV.43c) (left) and a pointlike vortex (IV.44b) (right).

Pointlike vortex

The "conjugate" flow to the previous one, i.e. that for which φ and ψ are exchanged, corresponds to the complex potential[(24)]

$$\phi(z) = \frac{\mathrm{i}\Gamma}{2\pi}\log(z-z_0) \quad \text{with} \quad \Gamma\in\mathbb{R},\ z_0\in\mathbb{C}. \tag{IV.44a}$$

Using as above polar coordinates (r,θ) centered on z_0, the flow velocity is purely tangential,

$$\vec{v}(r,\theta) = \frac{\Gamma}{2\pi r}\,\vec{e}_\theta, \tag{IV.44b}$$

as shown in Fig. IV.9 (right), where $\vec{e}_\theta$ is a unit orthoradial basis vector at (r,θ). The complex potential (IV.44a) thus describes a vortex situated at z_0.

In turn, the velocity potential and stream function read

$$\varphi(r,\theta) = -\frac{\Gamma}{2\pi}\,\theta, \quad \psi(r,\theta) = \frac{\Gamma}{2\pi}\log r, \tag{IV.44c}$$

to be compared with those for a flow source, Eq. (IV.43d).

Remark: When writing down the complex velocity potentials (IV.43a) or (IV.44a), we left aside the issue of the (logarithmic!) branch point at $z = z_0$—and we did not specify which branch of the logarithm we consider. Now, either potential corresponds to a flow that is actually defined on a doubly connected region, since the velocity diverges at $z = z_0$. From the discussion in § IV.4.2 b, on such domains the potential is a multivalued object, yet this is irrelevant for the physical quantities, namely the velocity field, which remains uniquely defined at each point. This is precisely what is illustrated here by the different branches of the logarithm, which differ by a constant integer multiple of $2\pi\mathrm{i}$ that does not affect the derivative.

Flow dipole

A further possible irrotational and incompressible two-dimensional flow is that defined by the complex potential

$$\phi(z) = \frac{\mu\,\mathrm{e}^{\mathrm{i}\alpha}}{z - z_0} \quad \text{with} \quad \mu \in \mathbb{R},\ \alpha \in \mathbb{R},\ z_0 \in \mathbb{C} \tag{IV.45a}$$

leading to the complex flow velocity

$$\mathsf{w}(z) = \frac{\mu\,\mathrm{e}^{\mathrm{i}\alpha}}{(z - z_0)^2}. \tag{IV.45b}$$

Again, both $\phi(z)$ and $\mathsf{w}(z)$ are singular at z_0.

Using polar coordinates (r, θ) centered on z_0, the flow velocity reads

$$\vec{\mathsf{v}}(r,\theta) = \frac{\mu}{r^2}\cos(\theta - \alpha)\,\vec{\mathrm{e}}_r + \frac{\mu}{r^2}\sin(\theta - \alpha)\,\vec{\mathrm{e}}_\theta, \tag{IV.45c}$$

which shows that the angle α gives the overall orientation of the flow with respect to the x-direction.

Setting for simplicity $\alpha = 0$ and coming back momentarily to Cartesian coordinates, the flow potential and stream function corresponding to Eq. (IV.45a) are

$$\varphi(x,y) = \frac{\mu x}{x^2 + y^2}, \quad \psi(x,y) = -\frac{\mu y}{x^2 + y^2}. \tag{IV.45d}$$

Thus, the streamlines are the curves $x^2 + y^2 = \text{const.} \times y$, i.e. they are circles centered on the y-axis

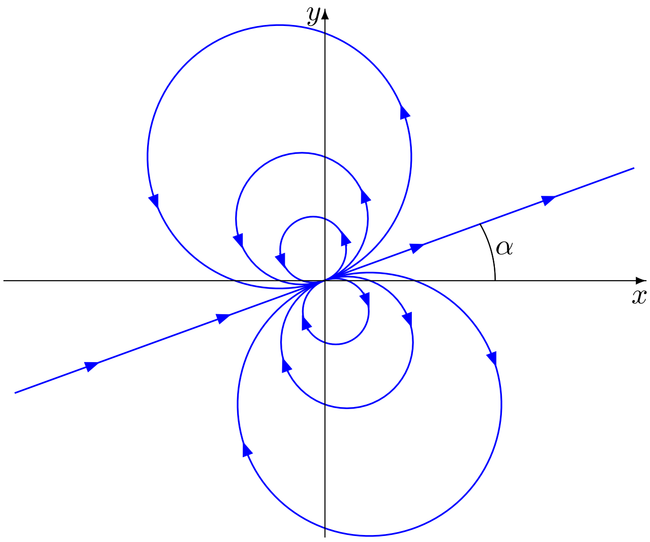

Figure IV.10 – Streamlines for a flow dipole (IV.45a) centered on the origin.

and tangent to the x-axis, as represented in Fig. IV.10, where everything is tilted by an angle α.

One can check that the flow dipole (IV.45a) is actually the superposition of a pair of infinitely close source and sink with the same mass flow rate in absolute value:

$$\phi(z) = \lim_{\varepsilon\to 0} \frac{\mu}{2\varepsilon}\left[\log\left(z - z_0 + \varepsilon\,\mathrm{e}^{-\mathrm{i}\alpha}\right) - \log\left(z - z_0 - \varepsilon\,\mathrm{e}^{-\mathrm{i}\alpha}\right)\right]. \tag{IV.46}$$

This is clearly fully analogous to an electric dipole potential being the superposition of the potentials created by electric charges $+q$ and $-q$—and justifies the denomination “dipole flow”.

One can similarly define higher-order multipoles: flow quadrupoles, octupoles, ..., for which the order of the pole of the velocity at z_0 increases (order 1 for a source or a sink, order 2 for a dipole, order 3 for a quadrupole, and so on).

Remarks:

$*$ The complex flow potentials considered until now—namely those of uniform flows (IV.42), sources or sinks (IV.43a), pointlike vortices (IV.44a), and dipoles (IV.45a) or multipoles—and their superpositions are the only two-dimensional flows valid on an unbounded domain.

As a matter of fact, demanding that the flow velocity $\vec{\mathsf{v}}(\vec{r})$ should be uniform at infinity and the complex velocity $\mathsf{w}(z)$ analytic except at a finite number of singularities—say only one, at z_0, to simplify the argumentation—, then $\mathsf{w}(z)$ may be expressed as a superposition of integer powers of $1/(z-z_0)$:

$$\mathsf{w}(z) = \sum_{p=0}^{\infty} \frac{a_{-p}}{(z-z_0)^p}, \tag{IV.47a}$$

since any positive power of $(z-z_0)$ would be unbounded when $|z| \to \infty$. Integrating over z, see Eq. (IV.41), the allowed complex potentials are of the form

$$\phi(z) = -a_0 z - a_{-1}\log(z-z_0) + \sum_{p=1}^{\infty} \frac{p\, a_{-p-1}}{(z-z_0)^p}. \tag{IV.47b}$$

$*$ Conversely, the reader can check—by computing the integral of $\mathsf{w}(z)$ along a contour at infinity—that the total mass flow rate and circulation of the velocity field for a given flow are respectively the real and imaginary parts of the residue a_{-1} in the Laurent series of its complex velocity $\mathsf{w}(z)$, i.e. are entirely governed by the source/sink term (IV.43a) and vortex term (IV.44a) in the complex potential.

$*$ Eventually, the singularities that arise in the flow velocity will in practice not be a problem, since these points will not be part of the physical flow, as we shall see on an example in § IV.4.3 c.

Flow inside or around a corner

As a last example, consider the complex flow potential

$$\phi(z) = A\,\mathrm{e}^{-\mathrm{i}\alpha}(z-z_0)^n \quad \text{with} \quad A \in \mathbb{R},\ \alpha \in \mathbb{R},\ n \geq \frac{1}{2},\ z_0 \in \mathbb{C}. \tag{IV.48a}$$

Except in the case $n = 1$, this potential cannot represent a flow on an unbounded domain, since one easily checks that the velocity is unbounded as $|z|$ goes to infinity. The interest of this potential lies rather the behavior in the vicinity of $z = z_0$.

As a matter, writing down the flow potential and the stream function in a system of polar coordinates centered on z_0,

$$\varphi(r,\theta) = A\,r^n \cos(n\theta - \alpha), \quad \psi(r,\theta) = A\,r^n \sin(n\theta - \alpha) \tag{IV.48b}$$

shows that they both are (π/n)-periodic functions of the polar angle θ. Thus the flow on the domain $\mathscr{D}$ delimited by the streamlines $\psi(r,\alpha)$ and $\psi(r,\alpha+\pi/n)$ is isolated from the motion in the remainder of the complex plane. One may therefore assume that there are walls along these two streamlines, and that the complex potential (IV.48a) describes a flow between them.

For $n = 1$, one recovers the uniform flow (IV.42)—in which we are free to put a wall along any streamline, restricting the domain $\mathscr{D}$ to a half plane instead of the whole plane. If $n > 1$, π/n is smaller than π and the domain $\mathscr{D}$ is comprised between a half-plane; in that case, the fluid motion is a flow *inside* a corner. On the other hand, for $\frac{1}{2} \leq n < 1$, $\pi/n > \pi$, so that the motion is a flow *past* a corner.

The streamlines for the flows obtained with six different values for n are displayed in Fig. IV.11, namely two flows in corners with angles $\pi/3$ and $2\pi/3$, a uniform flow in the upper half plane, two

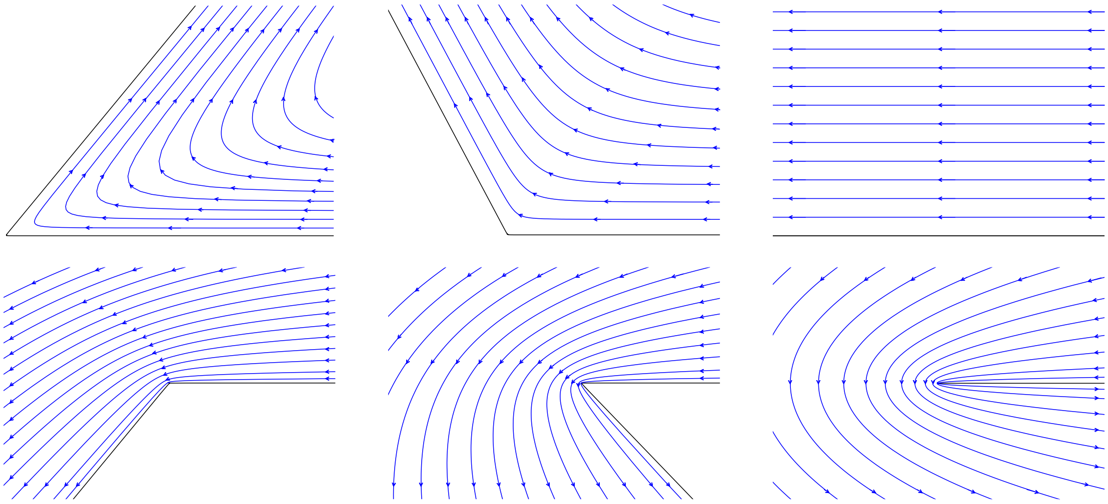

Figure IV.11 – Streamlines for the flow defined by potential (IV.48a) with from top to bottom and from left to right $n = 3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}$ and $\frac{1}{2}$.

flows past corners with inner angles $2\pi/3$ and $\pi/3$, and a flow past a flat plaque, corresponding respectively to $n = 3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}$ and $\frac{1}{2}$.

IV.4.3 c Two-dimensional flows past a cylinder

Thanks to the linearity of the Laplace differential equations, one may add "elementary" solutions of the previous paragraph to obtain new solutions, which describe possible two-dimensional flows. We now present two examples, which represent flows coming from infinity, where they are uniform, and falling on a cylinder—either immobile or rotating around its axis.

Acyclic flow

Let us superpose the complex potentials for a uniform flow (IV.42) along the x-direction and a flow dipole (IV.45a) situated at the origin and making an angle $\alpha = \pi$ with the vector $\vec{e}_x$:

$$\phi(z) = -\mathsf{v}_\infty\left(z + \frac{R^2}{z}\right), \tag{IV.49a}$$

where the dipole strength μ is written as $R^2\mathsf{v}_\infty$. Adopting polar coordinates (r, θ), this ansatz leads to the velocity potential and stream function

$$\varphi(r,\theta) = -\mathsf{v}_\infty\left(r + \frac{R^2}{r}\right)\cos\theta, \quad \psi(r,\theta) = -\mathsf{v}_\infty\left(r - \frac{R^2}{r}\right)\sin\theta. \tag{IV.49b}$$

One sees that the circle $r = R$ is a line of constant ψ, i.e. a streamline. This means that the flow outside that circle is decoupled from that inside. In particular, one may assume that the space inside the circle is filled by a solid obstacle, a "cylinder",[(25)] without changing the flow characteristics on $\mathbb{R}^2$ deprived from the disk $r < R$. The presence of this obstacle has the further advantage that it "hides" the singularity of the potential or the resulting velocity at $z = 0$, by cleanly removing it from the domain over which the flow is defined. This is illustrated, together with the streamlines for this flow, in Fig. IV.12.

From the complex potential (IV.49a) follows at once the complex velocity

$$\mathsf{w}(z) = \mathsf{v}_\infty\left(1 - \frac{R^2}{z^2}\right), \tag{IV.50a}$$

[(25)] The denomination is motivated by the fact that even though the flow characteristics depend on two spatial coordinates only, the actual flow might take in place in a three-dimensional space, in which case the obstacle is an infinite circular cylinder.

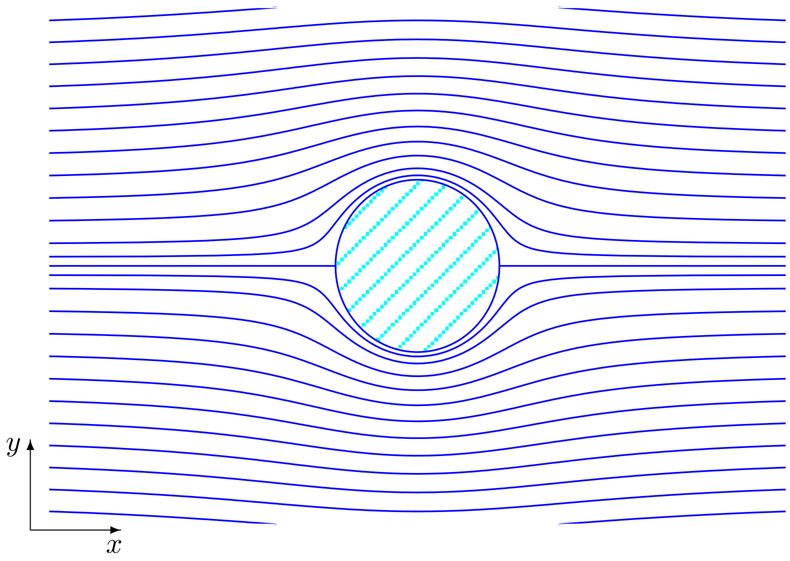

Figure IV.12 – Streamlines for the acyclic potential flow past a cylinder (IV.49a).

which in polar coordinates gives

$$\vec{\mathsf{v}}(r,\theta) = \mathsf{v}_\infty\left[\left(1-\frac{R^2}{r^2}\right)\cos\theta\,\vec{\mathrm{e}}_r - \left(1+\frac{R^2}{r^2}\right)\sin\theta\,\vec{\mathrm{e}}_\theta\right]. \qquad \text{(IV.50b)}$$

The latter is purely tangential for $r = R$, in agreement with the fact that the cylinder surface is a streamline. The flow velocity even fully vanishes at the points with $r = R$ and $\theta = 0$ or π, which are thus *stagnation points*.[(l)]

Assuming that the motion is stationary, one can calculate the force exerted on the cylinder by the flowing fluid. Invoking the Bernoulli equation (IV.11)—which holds since the flow is steady and incompressible—and using the absence of vorticity, which leads to the constant being the same throughout the flow, one obtains

$$\mathcal{P}(\vec{r}) + \frac{1}{2}\rho\vec{\mathsf{v}}(\vec{r})^2 = \mathcal{P}_\infty + \frac{1}{2}\rho\mathsf{v}_\infty^2,$$

where $\mathcal{P}_\infty$ denotes the pressure at infinity. That is, at each point on the surface of the cylinder

$$\mathcal{P}(R,\theta) = \mathcal{P}_\infty + \frac{1}{2}\rho\left[\mathsf{v}_\infty^2 - \vec{\mathsf{v}}(R,\theta)^2\right] = \mathcal{P}_\infty + \frac{1}{2}\rho\mathsf{v}_\infty^2\left(1-4\sin^2\theta\right),$$

where the second identity follows from Eq. (IV.50b). The resulting stress vector on the vector at a given θ is directed radially towards the cylinder center, $\vec{T}_s(R,\theta) = -\mathcal{P}(R,\theta)\,\vec{\mathrm{e}}_r(R,\theta)$. Integrating over $\theta \in [0, 2\pi]$, the total force on the cylinder due to the flowing fluid simply vanishes—in conflict with the intuition. This phenomenon is known as *d'Alembert paradox*.[(z)]

The intuition according to which the moving fluid should exert a force on the immobile obstacle is good. What we find here is a failure of the perfect-fluid model, which is in that case too idealized, by allowing the fluid to slip without friction along the obstacle.

Cyclic flow

To the flow profile which was just considered, we add a pointlike vortex (IV.44a) situated at the origin

$$\phi(z) = -\mathsf{v}_\infty\left(z+\frac{R^2}{z}\right) + \frac{\mathrm{i}\Gamma}{2\pi}\log\frac{z}{R}, \qquad \text{(IV.51a)}$$

(l) *Staupunkte*

(z) J. le Rond d'Alembert, 1717–1783

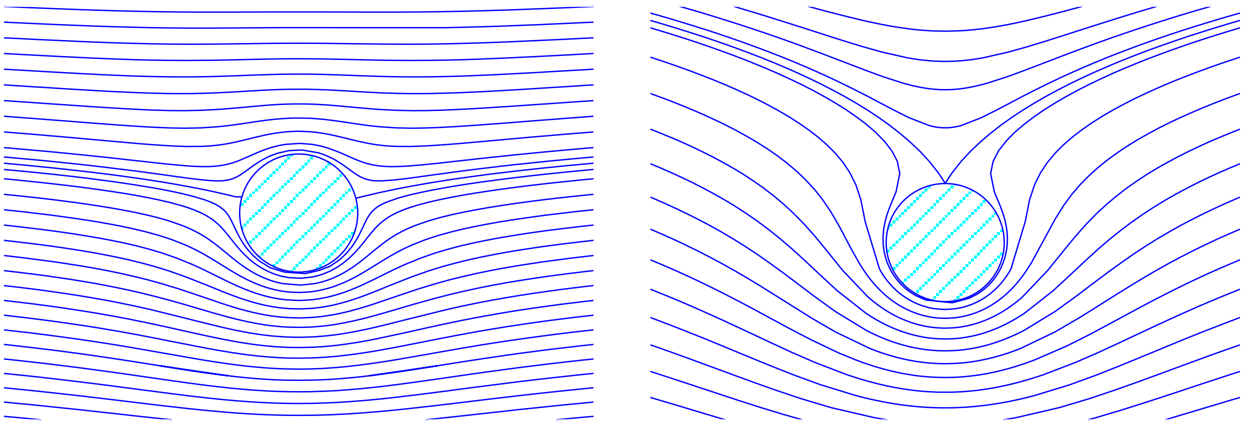

Figure IV.13 – Streamlines for the cyclic potential flow past a (rotating) cylinder (IV.51a) with $\Gamma/(4\pi R \mathsf{v}_\infty) = 0.25$ (left) or 1 (right).

where we have divided z by R in the logarithm to have a dimensionless argument, although this plays no role for the velocity. Comparing with the acyclic flow, which models fluid motion around a motionless cylinder, the complex potential may be seen as a model for the flow past a rotating cylinder, as in the case of the Magnus effect (§ IV.2.2 d).

Adopting polar coordinates (r,θ), the velocity potential and stream function read

$$\varphi(r,\theta) = -\mathsf{v}_\infty\left(r + \frac{R^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\,\theta, \quad \psi(r,\theta) = -\mathsf{v}_\infty\left(r - \frac{R^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\log\frac{r}{R}, \qquad \text{(IV.51b)}$$

so that the circle $r = R$ remains a streamline, delimiting a fixed obstacle.

The resulting velocity field reads in complex form

$$\mathsf{w}(z) = \mathsf{v}_\infty\left(1 - \frac{R^2}{z^2}\right) - \frac{\mathrm{i}\Gamma}{2\pi z}, \qquad \text{(IV.52a)}$$

and in polar coordinates

$$\vec{\mathsf{v}}(r,\theta) = \mathsf{v}_\infty\left[\left(1 - \frac{R^2}{r^2}\right)\cos\theta\,\vec{\mathrm{e}}_r - \left(1 + \frac{R^2}{r^2} - \frac{\Gamma}{2\pi r \mathsf{v}_\infty}\right)\sin\theta\,\vec{\mathrm{e}}_\theta\right]. \qquad \text{(IV.52b)}$$

The latter is again purely tangential for $r = R$, in agreement with the fact that the cylinder surface is a streamline.

One easily checks that when the strength of the vortex is not too large, namely $\Gamma \leq 4\pi R \mathsf{v}_\infty$, the flow has stagnations points on the surface of the cylinder—two if the inequality holds in the strict sense, a single degenerate point if $\Gamma = 4\pi R \mathsf{v}_\infty$—, as illustrated in Fig. IV.13. If $\Gamma > 4\pi R \mathsf{v}_\infty$, the flow defined by the complex potential (IV.51a) still has a stagnation point, yet now away from the surface of the rotating cylinder, as exemplified in Fig. IV.14.

In either case, repeating the same calculation based on the Bernoulli equation as for the acyclic flow allows one to derive the force exerted by the fluid on the cylinder. The resulting force no longer vanishes, but equals $-\Gamma\rho \mathsf{v}_\infty \vec{\mathrm{e}}_y$ on a unit length of the cylinder, where ρ is the mass density of the fluid and $\vec{\mathrm{e}}_y$ the unit basis vector in the y-direction. This is in line with the arguments presented in § IV.2.2 d.

IV.4.3 d Conformal deformations of flows

A further possibility to build two-dimensional potential flows is to "distort" the elementary solutions of § IV.4.3 b, or linear combinations of these building blocks. Such deformations may however not be arbitrary, since they must preserve the orthogonality at each point in the fluid of

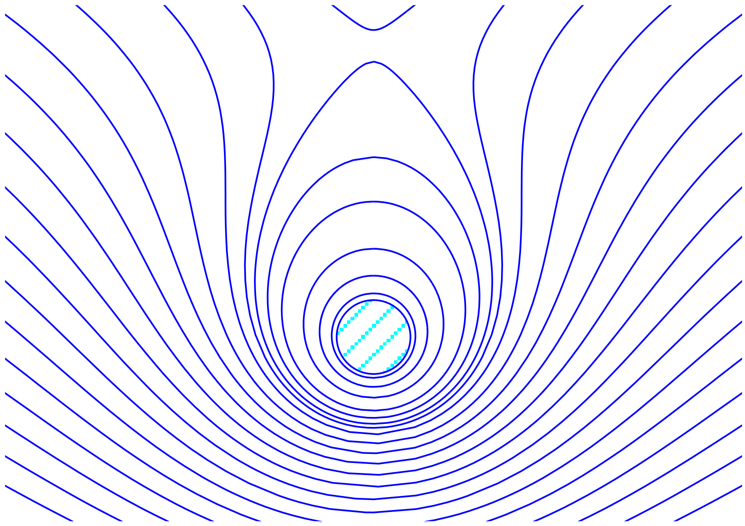

Figure IV.14 – Streamlines for the cyclic potential flow past a (rotating) cylinder (IV.51a) with $\Gamma/(4\pi R \mathsf{v}_\infty) = 4$.

the streamline (with constant ψ) and the equipotential line (constant φ) passing through that point. Besides rotations and dilations—which do not distort the profile of the solution, and are actually already taken into account in the solutions of § IV.4.3 b—, the generic class of transformations of the (complex) plane that preserve angles locally is that of *conformal maps*.

As recalled in Appendix C.4, such conformal mappings—between open subsets of the complex planes of variables z and Z—are defined by any holomorphic function $Z = f(z)$ whose derivative is everywhere non-zero and by its inverse F. If $\phi(z)$ denotes an arbitrary complex flow potential on the z-plane, then $\Phi(Z) \equiv \phi(F(Z))$ is a flow potential on the Z-plane. Applying the chain rule, the associated complex flow velocity is $\mathsf{w}(F(Z))F'(Z)$, where F' denotes the derivative of F.

A first example is to consider the trivial uniform flow with potential $\phi(z) = A\,z$, and the conformal mapping $z \mapsto Z = f(z) = z^{1/n}$ with $n \geq \frac{1}{2}$. The resulting complex flow potential on the Z-plane is $\Phi(Z) = -A\,Z^n$.

Except in the trivial case $n = 1$, $f(z)$ is singular at $z = 0$, where f' vanishes, so that the mapping is non-conformal: cutting a half-line ending at $z = 0$, f maps the complex plane deprived from this half-line onto an angular sector delimited by half-lines making an angle π/n—as already seen in § IV.4.3 b.

Joukowsky transform

A more interesting set of conformally deformed fluid flows consists of those provided by the use of the *Joukowsky transform*[(aa)]

$$Z = f(z) = z + \frac{R_J^2}{z} \qquad \text{(IV.53)}$$

where $R_J \in \mathbb{R}$.

The mapping (IV.53) is obviously holomorphic in the whole complex z-plane deprived of the origin—which a single pole—, and has 2 points $z = \pm R_J$ at which f' vanishes. These two singular points correspond in the Z-plane to algebraic branch points of the reciprocal function $z = F(Z)$ at $Z = \pm 2R_J$. To remove them, one introduces a branch cut along the line segment $|X| \leq 2R_J$. On the open domain $\mathcal{U}$ consisting of the complex Z-plane deprived from that line segment, F is holomorphic and conformal. One checks that the cut line segment is precisely the image by f of

(aa) Н. Е. Жуковский = N. E. Zhukovsky, 1847–1921 the circle $|z| = R_{\rm J}$ in the complex z-plane. Thus, f and F provide a bijective mapping between the exterior of the circle $|z| = R_{\rm J}$ in the z-plane and the domain $\mathcal{U}$ in the Z-plane.

Another property of the Joukowsky transform is that the singular points $z = \pm R_{\rm J}$ are zeros of f' of order 1, so that angles are locally multiplied by 2. That is, every continuously differentiable curve going through $z = \pm R_{\rm J}$ is mapped by f on a curve through $Z = \pm 2R_{\rm J}$ with an angular point, i.e. a discontinuous derivative, there.

Consider first the circle $\mathcal{C}(0,R)$ in the z-plane of radius $R > R_{\rm J}$ centered on the origin; it can be parameterized as

$$\mathcal{C}(0,R) = \big\{z = R\,{\rm e}^{{\rm i}\vartheta},\ 0 \le \vartheta \le 2\pi\big\}.$$

Its image in the Z-plane by the Joukowsky transform (IV.53) is the set of points such that

$$Z = \left(R + \frac{R_{\rm J}^2}{R}\right)\cos\vartheta + {\rm i}\left(R - \frac{R_{\rm J}^2}{R}\right)\sin\vartheta,\ \ 0 \le \vartheta \le 2\pi,$$

that is, the ellipse centered on the origin $Z = 0$ with semi-major resp. semi-minor axis $R + R_{\rm J}^2/R$ resp. $R - R_{\rm J}^2/R$ along the X- resp. Y-direction. **figure needed?** Accordingly, the flows past a circular cylinder studied in § IV.4.3 c can be deformed by f into flows past elliptical cylinders, where the angle between the ellipse major axis and the flow velocity far from the cylinder may be chosen at will.

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