IV.3 Vortex dynamics in perfect fluids

We now turn back to the case of an arbitrary flow $\vec{\mathsf{v}}(t,\vec{r})$, still in the case of a perfect fluid. The vorticity vector field, defined as the curl of the flow velocity field, was introduced in § II.1.2, together with the vorticity lines. Modulo a few assumptions on the fluid equation of state and the volume forces, one can show that vorticity is "frozen" in the flow of a perfect fluid, in the sense that the flux of vorticity across a material surface remains constant as the latter is being transported. This behavior will be investigated and formulated both at the integral level (§ IV.3.1) and differentially (§ IV.3.2).

IV.3.1 Circulation of the flow velocity. Kelvin's theorem

Definition: Let $\vec{\gamma}(t,\lambda)$ be a closed curve, parametrized by a real number $\lambda \in [0,1]$, which is being swept along by the fluid in its motion. The integral

$$\Gamma_{\vec{\gamma}}(t) \equiv \oint_{\vec{\gamma}} \vec{\mathsf{v}}(t,\vec{\gamma}(t,\lambda)) \cdot \mathrm{d}\vec{\ell} \tag{IV.13}$$

is called the *circulation* around the curve of the velocity field.

Remark: According to Stokes' theorem,[20] if the area bounded by the contour $\vec{\gamma}(t,\lambda)$ is simply connected, $\Gamma_{\vec{\gamma}}(t)$ equals the surface integral—the "flux"—of the vorticity field over every surface $\mathcal{S}_{\vec{\gamma}}(t)$ delimited by $\vec{\gamma}$:

$$\Gamma_{\vec{\gamma}}(t) = \int_{\mathcal{S}_{\vec{\gamma}}} \big[\vec{\nabla} \times \vec{\mathsf{v}}(t,\vec{r})\big] \cdot \mathrm{d}^2\vec{\mathcal{S}} = \int_{\mathcal{S}_{\vec{\gamma}}} \vec{\omega}(t,\vec{r}) \cdot \mathrm{d}^2\vec{\mathcal{S}}. \tag{IV.14}$$

Stated differently, the vorticity field is the flux density of the circulation of the velocity.

This relationship between circulation and vorticity will now be further exploited: we shall first establish and formulate results at the integral level, namely for the circulation, which will then be expressed at the differential level, i.e. in terms of the vorticity, in § IV.3.2.

Many results take a simpler form in a so-called *barotropic fluid*,[xlvi] in which the pressure can be expressed as function of only the mass density: $\mathcal{P} = \mathcal{P}(\rho)$, irrespective of whether the fluid is otherwise perfect or dissipative. An example of such a result is

Kelvin's circulation theorem:[u]

> *In a perfect barotropic fluid with conservative volume forces, the circulation of the flow velocity around a closed curve (enclosing a simply connected region) comoving with the fluid is conserved.* (IV.15a)

Denoting by $\vec{\gamma}(t,\lambda)$ the closed contour in the theorem,

$$\boxed{\frac{\mathrm{D}\Gamma_{\vec{\gamma}}(t)}{\mathrm{D}t} = 0.} \tag{IV.15b}$$

This result is also sometimes called *Thomson's theorem*.

Proof: For the sake of brevity, the arguments of the fields are omitted in case it is not necessary to specify them. Differentiating definition (IV.13) first gives

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\int_0^1 \frac{\partial\vec{\gamma}(t,\lambda)}{\partial\lambda} \cdot \vec{\mathsf{v}}(t,\vec{\gamma}(t,\lambda))\,\mathrm{d}\lambda = \int_0^1 \left[\frac{\partial^2\vec{\gamma}}{\partial\lambda\,\partial t} \cdot \vec{\mathsf{v}} + \frac{\partial\vec{\gamma}}{\partial\lambda} \cdot \left(\frac{\partial\vec{\mathsf{v}}}{\partial t} + \sum_i \frac{\partial\vec{\mathsf{v}}}{\partial x^i}\frac{\partial\gamma^i}{\partial t}\right)\right] \mathrm{d}\lambda.$$

Since the contour $\vec{\gamma}(t,\lambda)$ flows with the fluid, $\dfrac{\partial\vec{\gamma}(t,\lambda)}{\partial t} = \vec{\mathsf{v}}(t,\vec{\gamma}(t,\lambda))$, which leads to

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}}{\mathrm{D}t} = \int_0^1 \left\{\frac{\partial\vec{\mathsf{v}}}{\partial\lambda} \cdot \vec{\mathsf{v}} + \frac{\partial\vec{\gamma}}{\partial\lambda} \cdot \left[\frac{\partial\vec{\mathsf{v}}}{\partial t} + (\vec{\mathsf{v}} \cdot \vec{\nabla})\vec{\mathsf{v}}\right]\right\} \mathrm{d}\lambda.$$

[20] which in its classical form used here is also known as Kelvin–Stokes theorem...

[xlvi] *barotropes Fluid*

[u] W. Thomson, Baron Kelvin, 1824–1907

The first term in the curly brackets is clearly the derivative with respect to λ of $\vec{v}^2/2$, so that its integral along a closed curve vanishes. The second term involves the material derivative of $\vec{v}$, as given by the Euler equation. Using Eq. (III.19) with $\vec{a}_V = -\vec{\nabla}\Phi$ leads to

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}}{\mathrm{D}t} = \int_0^1 \left(-\frac{\vec{\nabla}\mathcal{P}}{\rho} - \vec{\nabla}\Phi\right) \cdot \frac{\partial\vec{\gamma}}{\partial\lambda}\,\mathrm{d}\lambda.$$

Again, the circulation of the gradient $\vec{\nabla}\Phi$ around a closed contour vanishes, leaving

$$\frac{\mathrm{D}\Gamma_{\vec{\gamma}}(t)}{\mathrm{D}t} = -\oint_{\vec{\gamma}} \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})}{\rho(t,\vec{r})} \cdot \mathrm{d}\vec{\ell}, \qquad \text{(IV.16)}$$

which constitutes the general case of Kelvin's circulation theorem for perfect fluids with conservative volume forces.
Transforming the contour integral with Stokes' theorem yields the surface integral of

$$\vec{\nabla} \times \left(\frac{\vec{\nabla}\mathcal{P}}{\rho}\right) = \frac{\vec{\nabla} \times \vec{\nabla}\mathcal{P}}{\rho} + \frac{\vec{\nabla}\mathcal{P} \times \vec{\nabla}\rho}{\rho^2} = \frac{\vec{\nabla}\mathcal{P} \times \vec{\nabla}\rho}{\rho^2}. \qquad \text{(IV.17)}$$

In a barotropic fluid, the rightmost term of this identity vanishes since $\vec{\nabla}\mathcal{P}$ and $\vec{\nabla}\rho$ are collinear, which yields relation (IV.15). □

Remark: Using relation (IV.14) and the fact that the area $S_{\vec{\gamma}}(t)$ bounded by the curve $\vec{\gamma}$ at time t defines a material surface, which will be transported in the fluid motion, Kelvin's theorem (IV.15) can be restated as

> *In a perfect barotropic fluid with conservative volume forces, the flux of the vorticity across a material surface is conserved.* (IV.18)

Kelvin's theorem leads to two trivial corollaries, namely

Helmholtz's theorem:[(v)]

> *In the flow of a perfect barotropic fluid with conservative volume forces, the vorticity lines and vorticity tubes move with the fluid.* (IV.19)

Similar to the definition of stream tubes in § I.3.3, a vorticity tube is defined as the surface formed by the vorticity lines tangent to a given closed geometrical curve.
In the case of vanishing vorticity $\vec{\omega} = \vec{0}$, one has

Lagrange's theorem:

> *In a perfect barotropic fluid with conservative volume forces, if the flow is irrotational at a given instant t, it remains irrotational at later times.* (IV.20)

Kelvin's circulation theorem (IV.15) and its corollaries imply that vorticity cannot be created nor destroyed in the flow of a perfect barotropic fluid with conservative volume forces. However, the more general form (IV.16) already show that in a non-barotropic fluid, there is a "source" for vorticity, which leads to the non-conservation of the circulation of the flow velocity. Similarly, non-conservative forces—for instance the Coriolis force in a rotating reference frame—may contribute a non-vanishing term in Eq. (IV.16) leading to a change in the circulation. We shall see in Sec. **??** that viscous stresses also affect the transport of vorticity in a fluid.

[(v)] H. von Helmholtz, 1821–1894

IV.3.2 Vorticity transport equation in perfect fluids

Consider the Euler equation (III.20), in the case of conservative volume forces, $\vec{a}_V = -\vec{\nabla}\Phi$. Taking the rotational curl of both sides yields after some straightforward algebra

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} - \vec{\nabla}\times\big[\vec{\mathsf{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})\big] = -\frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2}. \tag{IV.21}$$

This relation can be further transformed using the identity (we omit the variables)

$$\vec{\nabla}\times(\vec{\mathsf{v}}\times\vec{\omega}) = (\vec{\omega}\cdot\vec{\nabla})\vec{\mathsf{v}} + (\vec{\nabla}\cdot\vec{\omega})\vec{\mathsf{v}} - (\vec{\mathsf{v}}\cdot\vec{\nabla})\vec{\omega} - (\vec{\nabla}\cdot\vec{\mathsf{v}})\vec{\omega}.$$

Since the divergence of the vorticity field $\vec{\nabla}\cdot\vec{\omega}(t,\vec{r})$ vanishes, the previous two equations yield

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\omega}(t,\vec{r}) - \big[\vec{\omega}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}) = -\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2}. \tag{IV.22}$$

While it is tempting to introduce the material derivative $\mathrm{D}\vec{\omega}/\mathrm{D}t$ on the left hand side of this equation, for the first two terms, we rather define the whole left member to be a new derivative

$$\frac{\mathcal{D}_{\vec{\mathsf{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} \equiv \frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\omega}(t,\vec{r}) - \big[\vec{\omega}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}) \tag{IV.23a}$$

or equivalently

$$\frac{\mathcal{D}_{\vec{\mathsf{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} \equiv \frac{\mathrm{D}\vec{\omega}(t,\vec{r})}{\mathrm{D}t} - \big[\vec{\omega}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}). \tag{IV.23b}$$

We shall refer to $\mathcal{D}_{\vec{\mathsf{v}}}/\mathcal{D}t$ as the *comoving time derivative*, for reasons that will be explained at the end of this Section.

Using this definition, Eq. (IV.22) reads

$$\frac{\mathcal{D}_{\vec{\mathsf{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} = -\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}) - \frac{\vec{\nabla}\mathcal{P}(t,\vec{r})\times\vec{\nabla}\rho(t,\vec{r})}{\rho(t,\vec{r})^2}. \tag{IV.24}$$

In the particular case of a barotropic fluid—recall that we also assumed that it is ideal and only has conservative volume forces—this becomes

$$\frac{\mathcal{D}_{\vec{\mathsf{v}}}\vec{\omega}(t,\vec{r})}{\mathcal{D}t} = -\big[\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r})\big]\vec{\omega}(t,\vec{r}). \tag{IV.25}$$

Thus, the comoving time-derivative of the vorticity is parallel to itself.

From Eq. (IV.25), one deduces at once that if $\vec{\omega}(t,\vec{r})$ vanishes at some time t, it remains zero—which is the differential formulation of corollary (IV.20).

Invoking the continuity equation (III.9), the volume expansion rate $\vec{\nabla}\cdot\vec{\mathsf{v}}$ on the right hand side of Eq. (IV.25) can be replaced by $-(1/\rho)\mathrm{D}\rho/\mathrm{D}t$. For scalar fields, material derivative and comoving time-derivative coincide, which leads to the compact form

$$\frac{\mathcal{D}_{\vec{\mathsf{v}}}}{\mathcal{D}t}\left[\frac{\vec{\omega}(t,\vec{r})}{\rho(t,\vec{r})}\right] = \vec{0} \tag{IV.26}$$

for perfect barotropic fluids with conservative volume forces. That is, anticipating on the discussion of the comoving time derivative hereafter, $\vec{\omega}/\rho$ evolves in the fluid flow in the same way as the separation between two material neighboring points: the ratio is "frozen" in the fluid evolution.

Comoving time derivative

To understand the meaning of the comoving time derivative $\mathcal{D}_{\vec{\mathsf{v}}}/\mathcal{D}t$, let us come back to Fig. II.1 depicting the positions at successive times t and $t+\delta t$ of a small material vector $\delta\vec{\ell}(t)$. By definition of the material derivative, the change in $\delta\vec{\ell}$ between these two instants—as given by the trajectories of the two material points which are at $\vec{r}$ resp. $\vec{r}+\delta\vec{\ell}(t)$ at time t—is

$$\delta\vec{\ell}(t+\delta t)-\delta\vec{\ell}(t)=\frac{\mathrm{D}\delta\vec{\ell}(t)}{\mathrm{D}t}\delta t.$$

On the other hand, displacing the origin of $\delta\vec{\ell}(t+\delta t)$ to let it coincide with that of $\delta\vec{\ell}(t)$, one sees

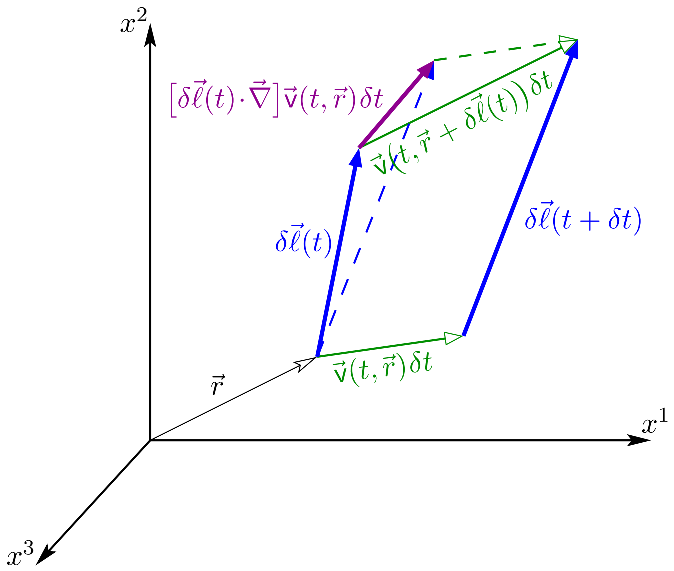

Figure IV.6 – Positions of a material line element $\delta\vec{\ell}$ at successive times t and $t+\delta t$.

on Fig. IV.6 that this change equals

$$\delta\vec{\ell}(t+\delta t)-\delta\vec{\ell}(t)=\big[\delta\vec{\ell}(t)\cdot\vec{\nabla}\big]\vec{v}(t,\vec{r})\delta t.$$

Equating both results and dividing by δt, one finds $\dfrac{\mathrm{D}\delta\vec{\ell}(t)}{\mathrm{D}t}=\big[\delta\vec{\ell}(t)\cdot\vec{\nabla}\big]\vec{v}(t,\vec{r})$, i.e. precisely

$$\frac{\mathcal{D}_{\vec{v}}\delta\vec{\ell}(t)}{\mathcal{D}t}=\vec{0}. \tag{IV.27}$$

Thus, the comoving time derivative of a material vector, which moves with the fluid, vanishes. In turn, the comoving time derivative at a given instant t of an arbitrary vector measures its rate of change with respect to a material vector with which it coincides at time t.

This interpretation suggests—this can be proven more rigorously—what the action of the comoving time derivative on a scalar field should be. In that case, $\mathcal{D}_{\vec{v}}/\mathcal{D}t$ should coincide with the material derivative, which already accounts for all changes—due to non-stationarity and convective transport—affecting material points in their motion. This justifies a posteriori our using $\mathcal{D}_{\vec{v}}\rho/\mathcal{D}t=\mathrm{D}\rho/\mathrm{D}t$ above.

More generally, the comoving time derivative introduced in Eq. (IV.23a) may be rewritten as

$$\frac{\mathcal{D}_{\vec{v}}}{\mathcal{D}t}(\,\cdot\,)\equiv\frac{\partial}{\partial t}(\,\cdot\,)+\mathcal{L}_{\vec{v}}(\,\cdot\,), \tag{IV.28}$$

where $\mathcal{L}_{\vec{v}}$ denotes the *Lie derivative* along the velocity field $\vec{v}(\vec{r})$, whose action on an arbitrary vector field $\vec{\omega}(\vec{r})$ is precisely (time plays no role here)

$$\mathcal{L}_{\vec{v}}\,\vec{\omega}(\vec{r})\equiv\big[\vec{v}(\vec{r})\cdot\vec{\nabla}\big]\vec{\omega}(\vec{r})-\big[\vec{\omega}(\vec{r})\cdot\vec{\nabla}\big]\vec{v}(\vec{r}),$$

while it operates on an arbitrary scalar field $\rho(\vec{r})$ according to

$$\mathcal{L}_{\vec{v}}\,\rho(\vec{r})\equiv\big[\vec{v}(\vec{r})\cdot\vec{\nabla}\big]\rho(\vec{r}).$$

More information on the Lie derivative, including its operation on 1-forms or more generally on $\binom{m}{n}$-tensors—from which the action of the comoving time derivative follows—, can be found e.g. in Ref. [17, Chap. 3.1–3.5].