Chapter IV

Non-relativistic flows of perfect fluids

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In the previous Chapter, we have introduced the coupled dynamical equations that govern the flows of perfect fluids in the non-relativistic regime, namely the continuity (III.9), Euler (III.18), and energy conservation (III.34) equations for the mass density $\rho(t,\vec{r})$, fluid velocity $\vec{\mathsf{v}}(t,\vec{r})$ and pressure $\mathcal{P}(t,\vec{r})$. The present Chapter discusses solutions of that system of equations, i.e. possible motions of perfect fluids,[(10)] obtained when using various assumptions to simplify the problem so as to render the equations tractable analytically.

In the simplest possible case, there is simply no motion at all in the fluid; yet the volume forces acting at each point still drive the behavior of the pressure and local mass density throughout the medium (Sec. IV.1). Steady flows, in which there is by definition no real dynamics, are also easily dealt with: both the Euler and energy conservation equations yield the Bernoulli equation, which can be further simplified by kinematic assumptions on the flow (Sec. IV.2).

Section IV.3 deals with the dynamics of vortices, i.e. of the vorticity vector field, in the motion of a perfect fluid. In such fluids, in case the pressure only depends on the mass density, there exists a quantity, related to vorticity, that remains conserved if the volume forces at play are conservative.

The latter assumption is also necessary to define potential flows (Sec. IV.4), in which the further hypothesis of an incompressible motion leads to simplified equations of motion, for which a number of exact mathematical results are known, especially in the case of two-dimensional flows.

Throughout the Chapter, it is assumed that the body forces in the fluid, whose volume density was denoted by $\vec{f}_V$ in Chapter III, are conservative, so that they derive from a potential. More specifically, anticipating the fact that these volume forces are proportional to the amount of mass

(10) ...at least in an idealized world. Yet the reader is encouraged to relate the results to observations of her everyday life—beyond the few illustrative examples provided by the author—, and to wonder how a small set of seemingly "simple" mathematical equations can describe a wide variety of physical phenomena.

they act upon, we introduce the potential energy per unit mass Φ, such that

$$\vec{f}_V(t,\vec{r}) = -\rho(t,\vec{r})\vec{\nabla}\Phi(t,\vec{r}). \tag{IV.1}$$

IV.1 Hydrostatics of a perfect fluid

The simplest possibility is that of *static* solutions of the system of equations governing the dynamics of perfect fluids, namely those with $\vec{v} = \vec{0}$ everywhere—in an appropriate global reference frame—and additionally $\partial/\partial t = 0$. Accordingly, there is strictly speaking no fluid flow: this is the regime of *hydrostatics*, for which the only[11] non-trivial equation—following from the Euler equation (III.18)—reads

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}). \tag{IV.2}$$

Throughout this Section, we adopt a fixed system of Cartesian coordinates $(x^1, x^2, x^3) = (x, y, z)$, with the basis vector $\vec{e}_3$ oriented vertically and pointing upwards. In the following examples, we shall consider the case of fluids in a homogeneous gravity field, leading to $\Phi(\vec{r}) = gz$, with $g = 9.8\ \mathrm{m\cdot s^{-2}}$.

Remark: If the stationarity condition is relaxed, the continuity equation still leads to $\partial\rho/\partial t = 0$, i.e. to a time-independent mass density. Whether time derivatives vanish or not makes no change in the Euler equation when $\vec{v} = \vec{0}$. Eventually, energy conservation requires that the internal energy density e—and thereby the pressure—follow the same time evolution as the "external" potential energy Φ. Thus, there is a non-stationary hydrostatics, but in which the time evolution decouples from the spatial problem.

IV.1.1 Archimedes' principle

Consider first a fluid, or a system of several fluids, at rest, occupying some region of space. Let $\mathcal{S}$ be a closed control surface inside that region, as depicted in Fig. IV.1 (left), and $\mathcal{V}$ be the volume delimited by $\mathcal{S}$. The fluid inside $\mathcal{S}$ will be denoted by Σ, and that outside by Σ'.

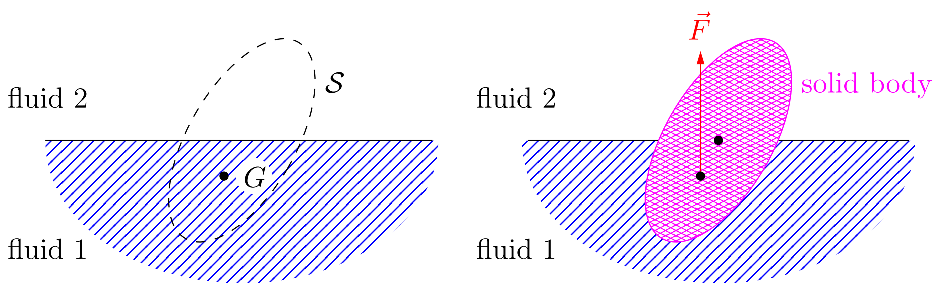

Figure IV.1 – Gedankenexperiment to illustrate Archimedes' principle.

The system Σ is in mechanical equilibrium, i.e. the sum of the gravity forces acting at each point of the volume $\mathcal{V}$ and the pressure forces exerted at each point of $\mathcal{S}$ by the fluid Σ' must vanish:

- The gravity forces at each point result in a single force $\vec{F}_G$, applied at the center of mass G of Σ, whose direction and magnitude are those of the weight of the system Σ.
- According to the equilibrium condition, the resultant of the pressure forces must equal $-\vec{F}_G$:

$$\oint_{\mathcal{S}} \mathcal{P}(\vec{r})\,\mathrm{d}^2\vec{\mathcal{S}} = -\vec{F}_G.$$

[11] This is true only in the case of perfect fluids; for dissipative ones, there emerge new possibilities, see § ??.

If one now replaces the fluid system Σ by a rigid body $\mathcal{B}$, while keeping the fluids Σ' outside $\mathcal{S}$ in the same equilibrium state, the mechanical stresses inside Σ' remain unchanged. Thus, the resultant of the contact forces exerted by Σ' on $\mathcal{B}$ is still given by $\vec{F} = -\vec{F}_G$, which still applies at the center of mass G of the fluid system Σ. This constitutes the celebrated *Archimedes*[(m)] *principle*:

$$\text{\textit{Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.}} \tag{IV.3}$$

In addition, we have obtained the point of application of the resultant force ("buoyancy"[(xlii)]) from the fluid.

Remark: If the center of mass G of the "displaced" fluid system does not coincide with the center of mass of the body $\mathcal{B}$, the latter will be submitted to a torque, since $\vec{F}$ and its weight are applied at two different points. This is e.g. the case in Fig. IV.1, which describes a mechanically unstable situation.

IV.1.2 Incompressible fluid

Consider first an incompressible fluid—or, more correctly, a fluid whose compressibility can as a first approximation be neglected—with constant, uniform mass density ρ.
The fundamental equation of hydrostatics (IV.2) in the uniform gravitational field $-g\,\vec{e}_z$ then yields

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial x} = \frac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0, \qquad \frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\rho g.$$

That is, one recovers Pascal's law[(n)]

$$\mathcal{P}(\vec{r}) = \mathcal{P}(z) = \mathcal{P}_0 - \rho g z, \tag{IV.4}$$

with $\mathcal{P}_0$ the pressure at the reference point with altitude $z = 0$.

For instance, the reader is possibly aware that at a depth of 10 meters under (liquid) water ($\rho = 10^3$ kg·m^{-3}), the pressure is

$$\mathcal{P}(-10\,\text{m}) = \mathcal{P}(0) + 10^3 \cdot g \cdot 10 \approx 2 \times 10^5 \text{ Pa},$$

with $\mathcal{P}(0) \approx 10^5$ Pa the typical atmospheric pressure at sea level.

IV.1.3 Fluid at global thermal equilibrium

To depart from the assumption of incompressibility, whose range of validity is quite limited, let us instead consider a fluid at global thermal equilibrium i.e. with a uniform temperature T; for instance, an ideal gas, obeying at each point the mechanical equation of state $\mathcal{P}(\vec{r}) = n(\vec{r})k_BT$, where n denotes the number density.

Denoting by m the mass of a molecule of that gas, the mass density is related to pressure and temperature by $\rho = m\mathcal{P}/k_BT$, so that Eq. (IV.2) reads

$$\frac{\partial\mathcal{P}(\vec{r})}{\partial x} = \frac{\partial\mathcal{P}(\vec{r})}{\partial y} = 0, \qquad \frac{\partial\mathcal{P}(\vec{r})}{\partial z} = -\frac{mg}{k_BT}\,\mathcal{P}(\vec{r}),$$

i.e. one obtains the *barometric formula*[(xliii)]

$$\mathcal{P}(\vec{r}) = \mathcal{P}(z) = \mathcal{P}_0 \exp\left(-\frac{mgz}{k_BT}\right).$$

[(xlii)] *statischer Auftrieb* [(xliii)] *barometrische Höhenformel*

[(m)] Archimedes, c.287–c.212 BC [(n)] B. Pascal, 1623–1662

Invoking the equation of state, one sees that the molecule number density $n(\vec{r})$ is also exponentially distributed, in agreement with the Maxwell distribution of statistical mechanics since mgz is the potential gravitational energy of a molecule at altitude z.

Taking as example air—which is a fictive ideal gas with molar mass[(12)] $\mathcal{N}_A\, m_{\rm air} = 29\ {\rm g\cdot mol^{-1}}$— the ratio $k_{\rm B}T/m_{\rm air}g$ equals 8.8×10^3 m for $T = 300$ K, i.e. the pressure drops by a factor 2 for every elevation gain of ca. 6 km. Obviously, however, assuming a constant temperature in the Earth atmosphere over such a length scale is unrealistic.

IV.1.4 Isentropic fluid

Let us now assume that the entropy per particle is constant throughout the perfect fluid at rest under study: $s/n =$ constant, with s the entropy density and n the particle number density.

> We shall show in § ?? that the ratio s/n is always conserved in the motion of a relativistic perfect fluid. Taking the low-velocity limit, one deduces the conservation of s/n in a non-relativistic non-dissipative flow: ${\rm D}(s/n)/{\rm D}t = 0$, implying that s/n is constant along pathlines, i.e. in the stationary regime along streamlines. Here we assume that s/n is constant everywhere.

Consider now the enthalpy $H = U + \mathcal{P}\mathcal{V}$ of the fluid, whose change in an infinitesimal process is the differential ${\rm d}H = T\,{\rm d}S + \mathcal{V}\,{\rm d}\mathcal{P} + \mu\,{\rm d}N$.[(13)] In this relation, μ denotes the chemical potential, which will however play no further role as we assume that the number of molecules in the fluid is constant, leading to ${\rm d}N = 0$. Dividing by N gives

$$ {\rm d}\!\left(\frac{H}{N}\right) = T\,{\rm d}\!\left(\frac{S}{N}\right) + \frac{\mathcal{V}}{N}\,{\rm d}\mathcal{P}, $$

or equivalently, in terms of the respective densities

$$ {\rm d}\!\left(\frac{w}{n}\right) = T\,{\rm d}\!\left(\frac{s}{n}\right) + \frac{1}{n}\,{\rm d}\mathcal{P}, $$

where w denotes the enthalpy density. Dropping the first term on the right-hand side, since s/n is assumed to be constant, and dividing by the mass of a molecule of the fluid, one finds

$$ {\rm d}\!\left(\frac{w}{\rho}\right) = \frac{1}{\rho}\,{\rm d}\mathcal{P}. \tag{IV.5} $$

This identity relates the change in enthalpy per unit mass w/ρ to the change in pressure $\mathcal{P}$ in an elementary isentropic process. If one considers a fluid at local thermodynamic equilibrium, in which w/ρ and $\mathcal{P}$ take different values at different places, the identity relates the difference in w/ρ to that in $\mathcal{P}$ between two (neighboring) points. Dividing by the distance between the two points, and in the limit where this distance vanishes, one derives an identity similar to (IV.5) with gradients instead of differentials:

$$ \vec{\nabla}\!\left[\frac{w(\vec{r})}{\rho(\vec{r})}\right] = \frac{1}{\rho(\vec{r})}\,\vec{\nabla}\mathcal{P}(\vec{r}) $$

Together with Eq. (IV.2), one thus obtains

$$ \vec{\nabla}\!\left[\frac{w(\vec{r})}{\rho(\vec{r})} + \Phi(\vec{r})\right] = \vec{0} \tag{IV.6} $$

that is $\dfrac{w(z)}{\rho(z)} + gz =$ constant.

[(12)] $\mathcal{N}_A$ denotes the Avogadro number.

[(13)] The reader in need of a short reminder on thermodynamics is referred to Appendix ??.

Taking as example an ideal diatomic gas, its internal energy is $U = \frac{5}{2}Nk_BT$, resulting in the enthalpy density

$$w = e + \mathcal{P} = \frac{5}{2}n k_B T + n k_B T = \frac{7}{2}n k_B T.$$

That is, $\frac{w}{\rho} = \frac{7}{2}\frac{k_B T}{m}$, with m the mass of a molecule of gas. Equation (IV.6) then gives

$$\frac{\mathrm{d}T(z)}{\mathrm{d}z} = -\frac{mg}{\frac{7}{2}k_B}.$$

In the case of air, the term on the right hand side equals $9.77 \times 10^{-3}\ \mathrm{K \cdot m^{-1}} = 9.77\ \mathrm{K \cdot km^{-1}}$, i.e. the temperature drops by ca. 10 degrees for an elevation gain of 1 km. This represents a much better modeling of the (lower) Earth atmosphere than the isothermal assumption of § IV.1.3.

Remarks:

∗ The *International Standard Atmosphere* (ISA)[14] model of the Earth atmosphere assumes a (piecewise) linear dependence of the temperature on the altitude. The adopted value of the temperature gradient in the troposphere is smaller than the above, namely $6.5\ \mathrm{K \cdot km^{-1}}$, to take into account the possible condensation of water vapor into droplets or even ice.

∗ Coming back to the derivation of relation (IV.6), if we had not assumed s/n constant, we would have found the relation

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = \vec{\nabla}\left[\frac{w(\vec{r})}{\rho(\vec{r})}\right] - T(\vec{r})\,\vec{\nabla}\left[\frac{s(\vec{r})}{\rho(\vec{r})}\right], \tag{IV.7}$$

which we shall use in § IV.2.1.

IV.2 Steady inviscid flows

We now turn to stationary solutions of the equations of motion for perfect fluids: all partial time derivatives vanish—and accordingly we shall drop the t variable—, yet the flow velocity $\vec{v}(\vec{r})$ may now be non-zero. Under those conditions, the equations (III.18) and (III.34) expressing the conservations of momentum and energy collapse onto a single equation (§ IV.2.1). Some applications of the latter in the particular case of an incompressible flow are then presented (§ IV.2.2).

IV.2.1 Bernoulli equation

Replacing in the Euler equation (III.20) the pressure term with the help of relation (IV.7) and the acceleration due to volume forces by its expression in terms of the potential energy per unit mass, one finds

$$\frac{\partial\vec{v}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{v}(t,\vec{r})^2}{2}\right] - \vec{v}(t,\vec{r}) \times \vec{\omega}(t,\vec{r}) = T(t,\vec{r})\,\vec{\nabla}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right] - \vec{\nabla}\left[\frac{w(t,\vec{r})}{\rho(t,\vec{r})}\right] - \vec{\nabla}\Phi(t,\vec{r}), \tag{IV.8}$$

which is rather more clumsy than the starting point (III.20), due to the many thermodynamic quantities it involves on its right hand side. Gathering all gradient terms together, one obtains

$$\frac{\partial\vec{v}(t,\vec{r})}{\partial t} + \vec{\nabla}\left[\frac{\vec{v}(t,\vec{r})^2}{2} + \frac{w(t,\vec{r})}{\rho(t,\vec{r})} + \Phi(t,\vec{r})\right] = \vec{v}(t,\vec{r}) \times \vec{\omega}(t,\vec{r}) + T(t,\vec{r})\,\vec{\nabla}\left[\frac{s(t,\vec{r})}{\rho(t,\vec{r})}\right]. \tag{IV.9}$$

In the stationary regime, the first term on the left-hand side disappears[15]—and we now omit the time variable from the equations.

[14] See e.g. https://en.wikipedia.org/wiki/International_Standard_Atmosphere.
[15] This yields a relation known as *Crocco's*[o] *theorem*[xliv]

[xliv] *Croccos Wirbelsatz*

[o] L. Crocco, 1909–1986

Let $\mathrm{d}\vec{\ell}(\vec{r})$ denote a vector tangential to the streamline at position $\vec{r}$, i.e. parallel to $\vec{\mathsf{v}}(\vec{r})$. In the scalar product of $\mathrm{d}\vec{\ell}(\vec{r})$ with Eq. (IV.9), both terms on the right hand side yield zero:

- the mixed product $\mathrm{d}\vec{\ell}(\vec{r})\cdot[\vec{\mathsf{v}}(\vec{r})\times\vec{\omega}(\vec{r})]$ is zero for it involves two collinear vectors;
- $\mathrm{d}\vec{\ell}(\vec{r})\cdot\vec{\nabla}[s(\vec{r})/\rho(\vec{r})]$ vanishes due to the conservation of s/n in flows of perfect fluids, which together with the stationarity reads $\vec{\mathsf{v}}(\vec{r})\cdot\vec{\nabla}[s(\vec{r})/n(\vec{r})]=0$, where n is proportional to ρ.

In turn, $\mathrm{d}\vec{\ell}(\vec{r})\cdot\vec{\nabla}$ represents the derivative along the direction of $\mathrm{d}\vec{\ell}(\vec{r})$, i.e. along the streamline at $\vec{r}$. Thus, the derivative of the term in squared brackets on the left hand side of Eq. (IV.9) vanishes along a streamline, i.e. the term remains constant on a streamline:

$$\frac{\vec{\mathsf{v}}(\vec{r})^2}{2}+\frac{w(\vec{r})}{\rho(\vec{r})}+\Phi(\vec{r}) = \text{constant along a streamline} \qquad \text{(IV.10)}$$

where the value of the constant depends on the streamline. Relation (IV.10) is referred to as the *Bernoulli equation*.[p]

In the stationary regime, the energy conservation equation (III.34), in which one recognizes the enthalpy density $w(\vec{r})=e(\vec{r})+\mathcal{P}(\vec{r})$ in the flux term, leads to the same relation (IV.10).

The first term in Eq. (III.34) vanishes due to the stationarity condition, leaving (we drop the variables)

$$\vec{\nabla}\cdot\left[\left(\frac{\vec{\mathsf{v}}^2}{2}+\frac{w}{\rho}+\Phi\right)\rho\,\vec{\mathsf{v}}\right]=0.$$

Applying the product rule to the left member, one finds a first term proportional to $\vec{\nabla}\cdot(\rho\,\vec{\mathsf{v}})$—which vanishes thanks to the continuity equation (III.9)—, leaving only the other term, which is precisely ρ times the derivative along $\vec{\mathsf{v}}$ of the left hand side of the Bernoulli equation. □

Bernoulli equation in particular cases

Coming back to Eq. (IV.9), if the steady flow is irrotational, i.e. $\vec{\omega}(\vec{r})=\vec{0}$ everywhere, and isentropic, i.e. $s(\vec{r})/n(\vec{r})$ is uniform, then the gradient on the left hand side vanishes. That is, the constant in the Bernoulli equation (IV.10) is independent of the streamline, i.e. it is the same everywhere.

In case the flow is incompressible, i.e. $\vec{\nabla}\cdot\vec{\mathsf{v}}(\vec{r})=0$, then the continuity equation shows that the mass density ρ becomes uniform throughout the fluid. One may then pull the factor $1/\rho$ inside the pressure gradient in the Euler equation (III.20). Repeating then the same steps as below Eq. (IV.9), one finds that the Bernoulli equation now reads

$$\text{In incompressible flows}\quad \frac{\vec{\mathsf{v}}(\vec{r})^2}{2}+\frac{\mathcal{P}(\vec{r})}{\rho}+\Phi(\vec{r})\quad\text{is constant along a streamline.} \qquad \text{(IV.11)}$$

This is the form which we shall use in the applications hereafter.

Can this form be reconciled with the other one (IV.10), which is still what follows from the energy conservation equation? Subtracting one from the other, one finds that the ratio $e(\vec{r})/\rho$ is constant along streamlines. That is, since ρ is uniform, the internal energy density is constant along pathlines—which coincide with streamlines in a steady flow. Now, thermodynamics expresses the differential $\mathrm{d}e$ through $\mathrm{d}s$ and $\mathrm{d}n$: since both entropy and particle number are conserved along a pathline, so is internal energy, i.e. Eq. (IV.10) is compatible with Eq. (IV.11).

[p] D. Bernoulli, 1700–1782

IV.2.2 Applications of the Bernoulli equation

Throughout this Section, we assume that the flow is incompressible, i.e. the mass density is uniform, and rely on Eq. (IV.11). Of course, one may release this assumption, in which case one should replace pressure by enthalpy density everywhere below.[16]

IV.2.2 a Drainage of a vessel. Torricelli's law

Consider a liquid contained in a vessel with a small hole at its bottom, through which the liquid can flow (Fig. IV.2).

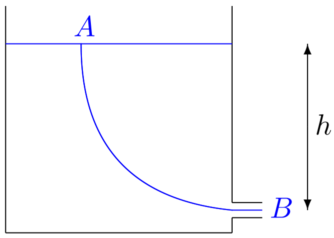

Figure IV.2

At points A and B that lie on the same streamline, the pressure in the liquid equals the atmospheric pressure[17] $\mathcal{P}_A = \mathcal{P}_B = \mathcal{P}_0$. The Bernoulli equation (at constant pressure) then yields

$$\frac{\mathsf{v}_A^2}{2} + gz_A = \frac{\mathsf{v}_B^2}{2} + gz_B,$$

with z_A resp. z_B the height of point A resp. B, i.e.

$$\mathsf{v}_B^2 = \mathsf{v}_A^2 + 2gh.$$

If the velocity at point A vanishes, one finds *Torricelli's law*[xlv][q]

$$\mathsf{v}_B = \sqrt{2gh}.$$

That is, the speed of efflux is the same as that acquired by a body *in free fall* from the same height h in the same gravity field.

Remark: To be allowed to apply the Bernoulli equation, one should first show that the liquid flows steadily. If the horizontal cross section of the vessel is much larger than the aperture of the hole and h large enough, this holds to a good approximation.

IV.2.2 b Venturi effect

Consider now the incompressible flow of a fluid inside the geometry illustrated in Fig. IV.3. As we shall only be interested in the average velocity or pressure of the fluid across a cross section of the tube, the flow is effectively one-dimensional.

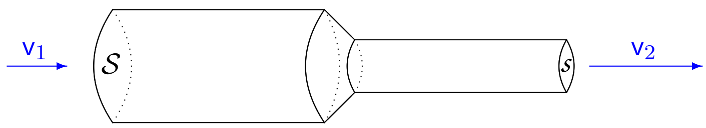

Figure IV.3

The conservation of the mass flow rate in the tube, which represents the integral formulation of the continuity equation (III.9), leads to $\rho\mathcal{S}\mathsf{v}_1 = \rho s \mathsf{v}_2$, i.e. $\mathsf{v}_2 = (\mathcal{S}/s)\mathsf{v}_1 > \mathsf{v}_1$, with $\mathcal{S}$ resp. s the area of the tube cross section in its broad resp. narrow section.

On the other hand, the Bernoulli equation at constant height, and thus potential energy, gives

$$\frac{\mathsf{v}_1^2}{2} + \frac{\mathcal{P}_1}{\rho} = \frac{\mathsf{v}_2^2}{2} + \frac{\mathcal{P}_2}{\rho}. \tag{IV.12}$$

All in all, the pressure in the narrow section is thus smaller than in the broad section, $\mathcal{P}_2 < \mathcal{P}_1$, which constitutes the *Venturi effect*.[r]

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[16] The author confesses that he has a better physical intuition of pressure than of enthalpy, hence his parti pris.

[17] One can show that the pressure in the liquid at point B equals the atmospheric pressure provided the local streamlines are parallel to each other—that is, if the flow is laminar.

[xlv] *Torricellis Theorem*

[q] E. Torricelli, 1608–1647 [r] G. B. Venturi, 1746–1822

Remarks:

∗ Using mass conservation and the Bernoulli equation, one can express v_1 or v_2 in terms of the tube cross section areas and the pressure difference. For instance, the mass flow rate reads

$$\rho\mathcal{S}\left[2\frac{\mathcal{P}_1-\mathcal{P}_2}{\rho}\bigg/\left(\frac{\mathcal{S}^2}{s^2}-1\right)\right]^{1/2}.$$

∗ From Eq. (IV.12) and the relation between the velocities, one sees that is the ratio $\mathcal{S}/s$, and thus v_2, is large enough, then $\mathcal{P}_2$ may be negative—which seems somewhat unsettling. Physically, the pressure does not truly become negative, but instead the assumed model breaks down. More precisely, when $\mathcal{P}_2$ reaches the saturated vapor pressure of the liquid, then the thermodynamically stable state is locally the gas phase, so that vapor bubbles will appear. This phenomenon is referred to as (hydrodynamic) *cavitation*.

IV.2.2 c Pitot tube

Figure IV.4 represents schematically the flow of a fluid around a (static) *Pitot tube*,(s) which is a device used to estimate a flow velocity through the measurement of a pressure difference. Three streamlines are shown, starting far away from the Pitot tube, where the flow is (approximately) uniform and has velocity $\vec{\mathsf{v}}$, which one wants to measure. The flow is assumed to be incompressible.

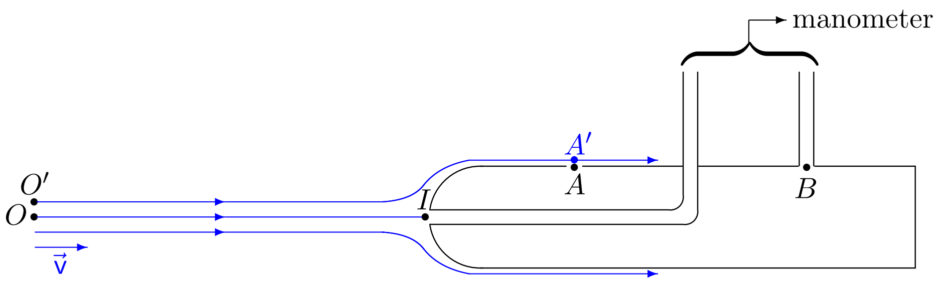

Figure IV.4 – Flow around a Pitot tube.

The Pitot tube consists of two long thin concentric tubes:

- Despite the presence of the hole at the end point I, the flow does not penetrate in the inner tube, so that $\vec{\mathsf{v}}_I = \vec{0}$ to a good approximation.
- In the broader tube, there is a hole at a point A, which is far enough from I to ensure that the fluid flow in the vicinity of A is no longer perturbed by the extremity of the tube: $\vec{\mathsf{v}}_A = \vec{\mathsf{v}}_{A'} \simeq \vec{\mathsf{v}}$, where the second identity holds thanks to the thinness of the tube, which thus perturbs the flow properties minimally. In addition, the pressure in the broader tube is uniform, so that $\mathcal{P}_A = \mathcal{P}_B$.

If one neglects the height differences—which is a posteriori justified by the numerical values we shall find—the (incompressible) Bernoulli equation gives first

$$\mathcal{P}_O + \rho\frac{\vec{\mathsf{v}}^2}{2} = \mathcal{P}_I$$

along the streamline OI, and

$$\mathcal{P}_{O'} + \rho\frac{\vec{\mathsf{v}}^2}{2} = \mathcal{P}_{A'} + \rho\frac{\vec{\mathsf{v}}_{A'}^2}{2}$$

along the streamline $O'A'$. Using $\mathcal{P}_{O'} \simeq \mathcal{P}_O$, $\mathcal{P}_{A'} \simeq \mathcal{P}_A$ and $\vec{\mathsf{v}}_{A'} \simeq \vec{\mathsf{v}}$, the latter identity leads to

(s) H. Pitot, 1695–1771

$\mathcal{P}_O \simeq \mathcal{P}_A = \mathcal{P}_B$. One thus finds

$$\mathcal{P}_I - \mathcal{P}_B = \rho \frac{\vec{v}^2}{2},$$

so that a measurement of $\mathcal{P}_I - \mathcal{P}_B$ gives an estimate of $|\vec{v}|$.

For instance, in air ($\rho \sim 1.3$ kg $\cdot$ m^{-3}) a velocity of 100 m $\cdot$ s^{-1} results in a pressure difference of 6.5×10^3 Pa $= 6.5 \times 10^{-2}$ atm. With a height difference h of a few centimeters between O and A', the neglected term $\rho g h$ is of order 0.1–1 Pa.

Remarks:

∗ The flow of a fluid with velocity $\vec{v}$ around a motionless Pitot tube is equivalent to the motion of a Pitot tube with velocity $-\vec{v}$ in a fluid at rest. Thus Pitot tubes are used to measure the speed of airplanes.[18]

∗ Is the flow of air really incompressible at velocities of 100 m $\cdot$ s^{-1} or higher? Not really, since the Mach number (II.18) becomes larger than 0.3. In practice, one thus rather uses the "compressible" Bernoulli equation (IV.10), yet the basic principles presented above remain valid.

IV.2.2 d Magnus effect

Consider an initially uniform and steady flow with velocity $\vec{v}_0$. One introduces in it a cylinder that rotates about its axis with angular velocity $\vec{\omega}_C$ perpendicular to the flow velocity (Fig. IV.5).

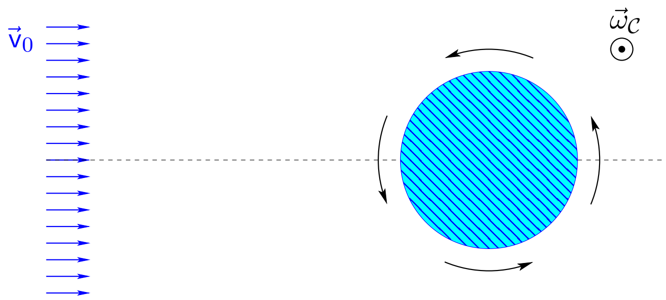

Figure IV.5 – Fluid flow around a rotating cylinder.

Intuitively, one can expect that the cylinder will drag the neighboring fluid layers along in its rotation.[19] In that case, the fluid velocity due to that rotation will add up to resp. be subtracted from the initial flow velocity in the lower resp. upper region close to the cylinder in Fig. IV.5.

Invoking now the Bernoulli equation—in which the height difference between both sides of the cylinder is neglected—, the pressure will be larger above the cylinder than below it. Accordingly, the cylinder will experience a resulting force directed downwards—more precisely, it is proportional to $\vec{v}_0 \times \vec{\omega}_C$—, which constitutes the *Magnus effect*.[t]

[18] When he introduced the idea in 1732, Pitot rather had the velocity of ships in his mind.

[19] Strictly speaking, this is not true in perfect fluids, only in real fluids with friction! Nevertheless, the tangential forces due to viscosity in the latter may be small enough that the Bernoulli equation remains approximately valid, as is assumed here.

[t] G. Magnus, 1802–1870