# Chapter II

# Kinematics of a continuous medium



The goal of fluid dynamics is to investigate the motion of fluids under consideration of the forces at play, as well as to study the mechanical stresses exerted by moving fluids on bodies with which they are in contact. The description of the motion itself, irrespective of the forces, is the object of *kinematics*.

The possibilities for the motion of a deformable continuous medium, in particular of a fluid, are richer than for a mere point particle or a rigid body: besides translations and global rotations, a deformable medium may also rotate locally and undergo... deformations! The latter term actually encompasses two different yet non-exclusive possibilities, namely either a change of shape or a variation of the volume. All these various types of motion are encoded in the local properties of the velocity field at each instant (Sec. II.1). Generic fluid motions are then classified according to several criteria, especially taking into account kinematics (Sec. II.2).

For the sake of reference, the characterization of deformations themselves, complementing that of their rate of change, is briefly presented in Sec. II.A. That formalism is not needed within fluid dynamics, but rather for the study of deformable solids, like elastic ones.

## II.1 Generic motion of a continuous medium

Let $\vec{\mathrm{v}}(t,\vec{r})$ denote the velocity field in a continuous medium, measured with respect to some reference frame $\mathcal{R}$. To illustrate (some of) the possible motions that occur in a deformable body, Fig. II.1 shows the positions at successive instants $t$ and $t+\delta t$ of a small "material vector" $\delta\vec{\ell}(t)$, that is, a continuous set of material points distributed along the straight line element stretching between two neighboring geometrical points. Let $\vec{r}$ and $\vec{r}+\delta\vec{\ell}(t)$ denote the geometrical endpoints of this material vector at time $t$.

Thanks to the continuity of the mappings $\vec{R}\mapsto\vec{r}(t,\vec{r})$ and its inverse $\vec{r}\mapsto\vec{R}(t,\vec{r})$, the material vector defined at instant $t$ remains a connected set of material points as time evolves, in particular at $t+\delta t$. Assuming that both the initial length $|\delta\vec{\ell}(t)|$ as well as $\delta t$ are small enough, the evolved set at $t+\delta t$ remains approximately along a straight line, and constitutes a new material vector, denoted $\delta\vec{\ell}(t+\mathrm{d}t)$. The position vectors of its endpoints simply follow from the initial positions of the corresponding material points: $\vec{r}$ resp. $\vec{r}+\delta\vec{\ell}(t)$, to which should be added the respective displacement vectors between $t$ and t+$\delta t$, namely the product by $\delta t$ of the initial velocity $\vec{\mathrm{v}}(t,\vec{r})$

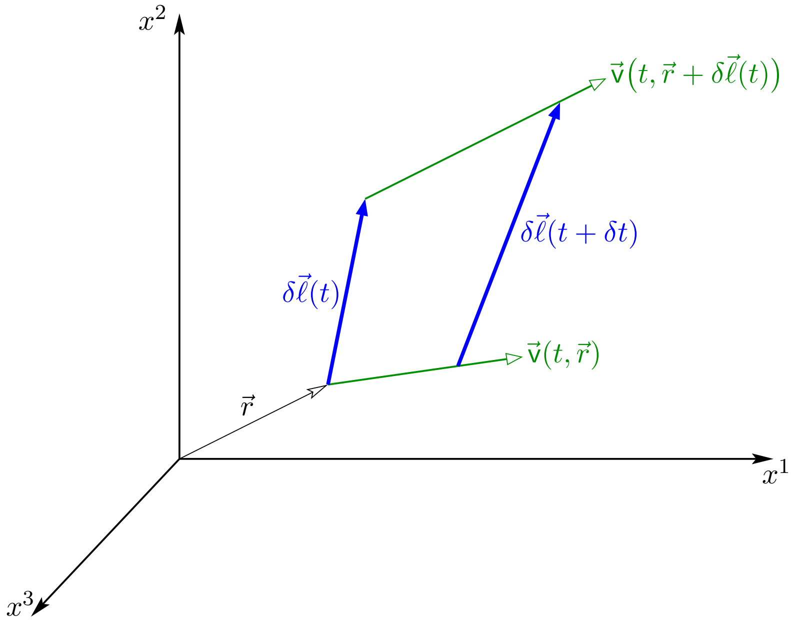


**Figure II.1** – Positions of a material line element $\delta\vec{\ell}$ at successive times $t$ and $t+\delta t$.

resp. $\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{\ell}(t)\big)$. That is, one finds

$$\delta\,\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \big[\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{\ell}(t)\big) - \vec{\mathsf{v}}\big(t,\vec{r}\big)\big]\,\delta t + \mathcal{O}\big(\delta t^2\big). \tag{II.1}$$

Figure II.1 already suggests that the motion of the material vector consists not only of a translation, but also of a rotation, as well as an "expansion"—the change in length of the vector.

## II.1.1 Local distribution of velocities in a continuous medium

Considering first a fixed time $t$, let $\vec{\mathsf{v}}(t,\vec{r})$ resp. $\vec{\mathsf{v}}(t,\vec{r})+\delta\vec{\mathsf{v}}$ be the velocity at the geometric point situated at position $\vec{r}$ resp. at $\vec{r}+\delta\vec{r}$ in $\mathcal{R}$.

Introducing for simplicity a system of Cartesian coordinates $(x^1,x^2,x^3)$ in $\mathcal{R}$, the Taylor expansion of the $i$-th component of the velocity field—which is at least piecewise $\mathscr{C}^1$ in its variables, see § I.3.2—gives to first order

$$\delta\mathsf{v}^i \simeq \sum_{j=1}^{3} \frac{\partial\mathsf{v}^i(t,\vec{r})}{\partial x^j}\,\delta x^j, \tag{II.2a}$$

where $\{\delta x^j\}$ denote the components of $\delta\vec{r}$. Introducing the $\binom{1}{1}$-tensor $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ whose components in the coordinate system used here are the partial derivatives $\partial\mathsf{v}^i(t,\vec{r})/\partial x^j$, the above relation can be recast in the coordinate-independent form

$$\delta\vec{\mathsf{v}} \simeq \vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\cdot\delta\vec{r}. \tag{II.2b}$$

Like every rank 2 tensor, the *velocity gradient tensor* $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$ at time $t$ and position $\vec{r}$ can be decomposed into the sum of the symmetric and an antisymmetric part:

$$\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) = \mathbf{D}(t,\vec{r}) + \mathbf{R}(t,\vec{r}), \tag{II.3a}$$

where one conventionally writes

$$\mathbf{D}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) + \big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big), \qquad \mathbf{R}(t,\vec{r}) \equiv \frac{1}{2}\Big(\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r}) - \big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}\Big) \tag{II.3b}$$

with $\big[\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})\big]^{\mathsf{T}}$ the transposed tensor to $\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}(t,\vec{r})$. These definitions are to be understood as follows:

Using the same Cartesian coordinate system as above, the components of the two tensors **D**, **R**, viewed for simplicity as $\binom{0}{2}$-tensors, respectively read

$$\mathsf{D}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right], \qquad \mathsf{R}_{ij}(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}_i(t,\vec{r})}{\partial x^j} - \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x^i}\right]. \tag{II.3c}$$

Note that here we have silently used the fact that for Cartesian coordinates, the position—subscript or superscript—of the index does not change the value of the component, i.e. numerically $\mathsf{v}_i = \mathsf{v}^i$ for every $i \in \{1,2,3\}$.

Relations (II.3c) clearly represent the desired symmetric and antisymmetric parts. However, one sees that the definitions would not appear to fulfill their task if the indices were not both either up or down, as e.g.

$$\mathsf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\left[\frac{\partial \mathsf{v}^i(t,\vec{r})}{\partial x^j} + \frac{\partial \mathsf{v}_j(t,\vec{r})}{\partial x_i}\right],$$

in which the symmetry is no longer obvious. The trick is to rewrite the previous identity as

$$\mathsf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}\delta^{ik}\delta^l{}_j\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right] = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\partial \mathsf{v}_k(t,\vec{r})}{\partial x^l} + \frac{\partial \mathsf{v}_l(t,\vec{r})}{\partial x^k}\right],$$

where we have used the fact that the metric tensor of Cartesian coordinates coincides with the Kronecker symbol. To fully generalize to curvilinear coordinates, the partial derivatives in the rightmost term should be replaced by the covariant derivatives discussed in Appendix **??**, leading eventually to

$$\mathsf{D}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} + \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4a}$$

$$\mathsf{R}^i{}_j(t,\vec{r}) = \frac{1}{2}g^{ik}(t,\vec{r})g^l{}_j(t,\vec{r})\left[\frac{\mathrm{d}\mathsf{v}_k(t,\vec{r})}{\mathrm{d}x^l} - \frac{\mathrm{d}\mathsf{v}_l(t,\vec{r})}{\mathrm{d}x^k}\right] \tag{II.4b}$$

With these new forms, which are valid in any coordinate system, the raising or lowering of indices does not affect the visual symmetric or antisymmetric aspect of the tensor.

Using the tensors **D** and **R** we just introduced, whose physical meaning will be discussed at length in § II.1.2–II.1.3, relation (II.2b) can be recast as

$$\vec{\mathsf{v}}\big(t,\vec{r}+\delta\vec{r}\big) = \vec{\mathsf{v}}\big(t,\vec{r}\big) + \mathsf{D}(t,\vec{r})\cdot\delta\vec{r} + \mathsf{R}(t,\vec{r})\cdot\delta\vec{r} + \mathcal{O}\big(|\delta\vec{r}|^2\big) \tag{II.5}$$

where as stated at the beginning every field is considered at the same time.

Under consideration of relation (II.5) with $\delta\vec{r} = \delta\vec{\ell}(t)$, Eq. (II.1) for the time evolution of the material line element becomes

$$\delta\vec{\ell}(t+\delta t) = \delta\vec{\ell}(t) + \big[\mathsf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathsf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t)\big]\delta t + \mathcal{O}\big(\delta t^2\big).$$

Subtracting $\delta\vec{\ell}(t)$ from both sides, dividing by $\delta t$ and taking the limit $\delta t \to 0$, one finds for the rate of change of the material vector:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathsf{D}(t,\vec{r})\cdot\delta\vec{\ell}(t) + \mathsf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t) \tag{II.6}$$

In the following two subsections, we shall investigate the physical content of each of the tensors $\mathsf{R}(t,\vec{r})$ and $\mathsf{D}(t,\vec{r})$.

### II.1.2 Rotation rate tensor and vorticity vector

The tensor $\mathbf{R}(t,\vec{r})$ defined by Eq. (II.3b) is called, for reasons that will become clearer below, *rotation rate tensor*.[(xxii)]

By construction, this tensor is antisymmetric. Accordingly, one can naturally associate with it a dual (pseudo)-vector $\vec{\Omega}(t,\vec{r})$, such that for any vector $\vec{V}$

$$\mathbf{R}(t,\vec{r}) \cdot \vec{V} = \vec{\Omega}(t,\vec{r}) \times \vec{V} \quad \forall \vec{V} \in \mathbb{R}^3. \tag{II.7}$$

In Cartesian coordinates, the components of $\vec{\Omega}(t,\vec{r})$ are related to those of the rotation rate tensor by

$$\Omega^i(t,\vec{r}) \equiv -\frac{1}{2}\sum_{j,k=1}^{3} \epsilon^{ijk}\,\mathbf{R}_{jk}(t,\vec{r}) \tag{II.8a}$$

with $\epsilon^{ijk}$ the totally antisymmetric Levi-Civita symbol. Using the antisymmetry of $\mathbf{R}(t,\vec{r})$, this equivalently reads

$$\Omega^1(t,\vec{r}) \equiv -\mathbf{R}_{23}(t,\vec{r}), \quad \Omega^2(t,\vec{r}) \equiv -\mathbf{R}_{31}(t,\vec{r}), \quad \Omega^3(t,\vec{r}) \equiv -\mathbf{R}_{12}(t,\vec{r}). \tag{II.8b}$$

Comparing with Eq. (II.3c), one finds

$$\vec{\Omega}(t,\vec{r}) = \frac{1}{2}\vec{\nabla} \times \vec{\mathsf{v}}(t,\vec{r}). \tag{II.9}$$

Proof of Eqs. (II.7), (II.9): introducing the Cartesian components $\{V^j\}$ of $\vec{V}$ and dropping for brevity the $(t,\vec{r})$-dependence of fields, the $i$-th component of $\mathbf{R}\cdot\vec{V}$ reads

$$\mathbf{R}_{ij}V^j = \frac{1}{2}(\partial_j \mathsf{v}_i - \partial_i \mathsf{v}_j)V^j,$$

where we used the summation convention over the repeated index $j$ and the shorthand notation $\partial_i$ for the partial derivative with respect to $x^i$. This may further be rewritten as

$$\mathbf{R}_{ij}V^j = -\frac{1}{2}(\delta_i^k\delta_j^l - \delta_j^k\delta_i^l)(\partial_k \mathsf{v}_l)V^j,$$

which now involves three sums. The term with the four Kronecker symbols is in fact the sum (over a fifth index $m$) of the product $\epsilon_{ijm}\epsilon^{mkl}$ of Levi-Civita symbols:

$$\mathbf{R}_{ij}V^j = -\frac{1}{2}\epsilon_{ijm}\epsilon^{mkl}(\partial_k \mathsf{v}_l)V^j.$$

On the right hand side of this identity, $\epsilon^{mkl}\partial_k \mathsf{v}_l$ is the $m$-th component of the curl $\vec{\nabla}\times\vec{\mathsf{v}}$, i.e. using definition (II.9):

$$\mathbf{R}_{ij}V^j = -\epsilon_{ijm}\Omega^m V^j = \epsilon_{imj}\Omega_m V^j,$$

which is precisely the $i$-th component of $\Omega \times \vec{\mathsf{v}}$. □

Let us now rewrite relation (II.6) with the help of the vector $\vec{\Omega}(t,\vec{r})$, assuming that $\mathbf{D}(t,\vec{r})$ vanishes so as to isolate the effect of the remaining term. Under this assumption, the rate of change of the material vector between two neighboring points reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{\ell}(t) = \mathbf{R}(t,\vec{r})\cdot\delta\vec{\ell}(t) = \vec{\Omega}(t,\vec{r}) \times \delta\vec{\ell}(t). \tag{II.10}$$

The term on the right hand side is then exactly the rate of rotation of a vector $\delta\vec{\ell}(t)$ in the motion of a rigid body with instantaneous angular velocity $\vec{\Omega}(t,\vec{r})$. Accordingly, the pseudovector $\vec{\Omega}(t,\vec{r})$ is referred to as *local angular velocity*.[(xxiii)] This a posteriori justifies the denomination *rotation rate tensor* for the antisymmetric tensor $\mathbf{R}(t,\vec{r})$.

(xxii) *Wirbeltensor* (xxiii) *Wirbelvektor*

**Remarks:**

∗ Besides the local angular velocity $\vec{\Omega}(t,\vec{r})$, one also defines the *vorticity vector*[(xxiv)] as the curl of the velocity field

$$\boxed{\vec{\omega}(t,\vec{r}) \equiv \vec{\nabla} \times \vec{\mathrm{v}}(t,\vec{r})} = 2\vec{\Omega}(t,\vec{r}). \tag{II.11}$$

In fluid mechanics, the vorticity is actually more often used than the local angular velocity.

∗ The local angular velocity $\vec{\Omega}(t,\vec{r})$ or equivalently the vorticity vector $\vec{\omega}(t,\vec{r})$ define, at fixed $t$, divergence-free (pseudo)vector fields, since obviously $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathrm{v}}) = 0$. The corresponding field lines are called *vorticity lines*[(xxv)] and are given by [cf. Eq. (I.15)]

$$\mathrm{d}\vec{x} \times \vec{\omega}(t,\vec{r}) = \vec{0} \tag{II.12a}$$

or equivalently, at a point where none of the components of the vorticity vector vanishes,

$$\frac{\mathrm{d}x_1}{\omega^1(t,\vec{r})} = \frac{\mathrm{d}x_2}{\omega^2(t,\vec{r})} = \frac{\mathrm{d}x_3}{\omega^3(t,\vec{r})}. \tag{II.12b}$$

(xxiv) *Wirbligkeit* (xxv) *Wirbellinien*