I.4 Mechanical stress

I.4.1 Forces in a continuous medium

Consider a closed material domain $\mathcal{V}$ inside the volume $\mathcal{V}_t$ occupied by a continuous medium, and let $\mathcal{S}$ denote the (geometric) surface enclosing $\mathcal{V}$. One distinguishes between two classes of forces acting on this domain:

- *Volume* or *body forces*,[(xiv)] which act at each point of the bulk volume of $\mathcal{V}$.
  Examples are weight, long-range electromagnetic forces or, in non-inertial reference frames, fictitious forces (Coriolis, centrifugal).
  For such forces, which tend to be proportional to the volume they act on, it will later be more convenient to introduce the corresponding volumic force density.
- *Surface* or *contact forces*,[(xv)] which act on the surface $\mathcal{S}$, like friction. These will be now discussed in further detail.

Consider an infinitesimally small geometrical surface element $\mathrm{d}^2\mathcal{S}$ at point P. Let $\mathrm{d}^2\vec{F}_s$ denote the surface force through $\mathrm{d}^2\mathcal{S}$. That is, $\mathrm{d}^2\vec{F}_s$ is the contact force, due to the medium exterior to $\mathcal{V}$, that a "test" material surface coinciding with $\mathrm{d}^2\mathcal{S}$ would experience. The vector

$$\vec{T}_{\mathrm{s}} \equiv \frac{\mathrm{d}^2\vec{F}_s}{\mathrm{d}^2\mathcal{S}}, \tag{I.20}$$

representing the surface density of contact forces, is called *(mechanical) stress vector*[(xvi)] on $\mathrm{d}^2\mathcal{S}$.

[(xiv)] *Volumenkräfte*

[(xv)] *Oberflächenkräfte*

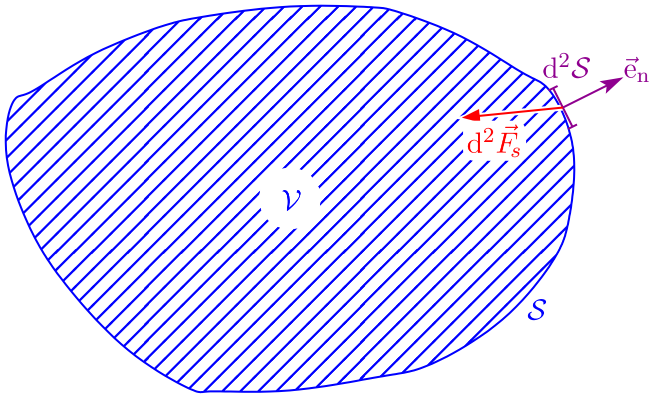

Figure I.2

The corresponding unit in the SI system is the Pascal, with 1 Pa $= 1\,\mathrm{N}\cdot\mathrm{m}^{-2}$.

Purely geometrically, the stress vector $\vec{T}_s$ on a given surface element $\mathrm{d}^2\mathcal{S}$ at a given point can be decomposed into two components, namely

- a vector orthogonal to the plane tangent at P to $\mathrm{d}^2\mathcal{S}$, the so-called *normal stress*[(xvii)]; when it is directed towards the interior resp. exterior of the medium domain being acted on, it also referred to as *compression*[(xviii)] resp. *tension*[(xix)];
- a vector in the tangent plane in P, called *shear stress*[(xx)] and often denoted as $\vec{\tau}$.

Despite the short notation adopted in Eq. (I.20), the stress vector depends not only on the position of the geometrical point P where the infinitesimal surface element $\mathrm{d}^2\mathcal{S}$ lies, but also on the orientation of the surface. Let $\vec{e}_n$ denote the normal unit vector to the surface element, directed towards the exterior of the volume $\mathcal{V}$ (cf. Fig. I.2), and let $\vec{r}$ denote the position vector of P in a given reference frame. The relation between $\vec{e}_n$ and the stress vector $\vec{T}_s$ on $\mathrm{d}^2\mathcal{S}$ is then linear:

$$\vec{T}_s = \boldsymbol{\sigma}(\vec{r}) \cdot \vec{e}_n, \tag{I.21a}$$

with $\boldsymbol{\sigma}(\vec{r})$ a symmetric tensor of rank 2, the so-called (*Cauchy*[(f)]) *stress tensor*.[(xxi)]

In a given coordinate system, relation (I.21a) yields

$$T_s^i = \sum_{j=1}^{3} \boldsymbol{\sigma}^i{}_j\, e_n^j \tag{I.21b}$$

with T_s^i resp. e_n^j the coordinates of the vectors $\vec{T}_s$ resp. $\vec{e}_n$, and $\boldsymbol{\sigma}^i{}_j$ the $\binom{1}{1}$-components of the stress tensor.

While valid in the case of a three-dimensional position space, equation (I.21a) should actually be better formulated to become valid in arbitrary dimension. Thus, the unit-length “normal vector” to a surface element at point P is rather a *1-form* acting on the vectors of the tangent space to the surface at P. As such, it should be represented as the transposed of a vector $[(\vec{e}_n)^\mathsf{T}]$, which multiplies the stress tensor from the left:

$$\boxed{\vec{T}_s = (\vec{e}_n)^\mathsf{T} \cdot \boldsymbol{\sigma}(\vec{r}).} \tag{I.21c}$$

(xvi) *Mechanischer Spannungsvektor* (xvii) *Normalspannung* (xviii) *Druckspannung* (xix) *Zugsspannung* (xx) *Scher-, Tangential-* oder *Schubspannung* (xxi) (*Cauchy'scher*) *Spannungstensor*

(f) A.L. Cauchy, 1789–1857

This shows that the Cauchy stress tensor is a $\binom{2}{0}$-tensor (a "bivector"), which maps 1-forms onto vectors. In terms of coordinates, this gives, using Einstein's summation convention

$$T_s^j = \mathrm{e}_{\mathrm{n},i}\, \boldsymbol{\sigma}^{ij}\,, \tag{I.21d}$$

which thanks to the symmetry of $\boldsymbol{\sigma}$ is equivalent to the relation given above.

Remark: The symmetry property of the Cauchy stress tensor is intimately linked to the assumption that the material points constituting the continuous medium have no intrinsic angular momentum.

I.4.2 Fluids

With the help of the notion of mechanical stress, we may now introduce the definition of a *fluid*, which is the class of continuous media whose motion is described by hydrodynamics:

A fluid is a continuous medium that deforms itself as long as it is submitted to shear stresses. (I.22)

Turning this definition around, one sees that in a fluid *at rest*—or, to be more accurate, studied in a reference frame with respect to which it is at rest—the mechanical stresses are necessarily normal. That is, the stress tensor is in each point diagonal.

More precisely, for a locally isotropic fluid—which means that the material points are isotropic, which is the case throughout these notes—the stress $\binom{2}{0}$-tensor is everywhere proportional to the inverse metric tensor:

$$\boldsymbol{\sigma}(t,\vec{r}) = -\mathcal{P}(t,\vec{r})\, \mathbf{g}^{-1}(t,\vec{r}) \tag{I.23}$$

with $\mathcal{P}(t,\vec{r})$ the *hydrostatic pressure* at position $\vec{r}$ at time t.

Going back to relation (I.21b), the stress vector will be parallel to the "unit normal vector" in any coordinate system if the square matrix of the $\binom{1}{1}$-components $\boldsymbol{\sigma}^i{}_j$ is proportional to the identity matrix, i.e. $\boldsymbol{\sigma}^i{}_j \propto \delta^i{}_j$, where we have introduced the Kronecker symbol. To obtain the $\binom{2}{0}$-components $\boldsymbol{\sigma}^{ik}$, one has to multiply $\boldsymbol{\sigma}^i{}_j$ by the component g^{jk} of the inverse metric tensor, summing over k, which precisely gives Eq. (I.23).

Remarks:

∗ Definition (I.22), as well as the two remarks hereafter, rely on an intuitive picture of "deformations" in a continuous medium. To support this picture with some mathematical background, we shall introduce in Sec. **??** an appropriate strain tensor, which quantifies these deformations, at least as long as they remain small.

∗ A deformable solid will also deform itself when submitted to shear stress! However, for a given fixed amount of tangential stress, the solid will after some time reach a new, deformed equilibrium position—otherwise, it is not a solid, but a fluid.

∗ The previous remark is actually a simplification, valid on the typical time scale of human beings. Thus, materials which in our everyday experience are solids—as for instance those forming the mantle of the Earth—will behave on a longer time scale as fluids—in the previous example, on geological time scales. Whether a given substance behaves as a fluid or a deformable solid is sometimes characterized by the dimensionless *Deborah number* [9], which compares the typical time scale for the response of the substance to a mechanical stress and the observation time.

∗ Even nicer, the fluid vs. deformable solid behavior may actually depend on the intensity of the applied shear stress: ketchup!