Tutorial sheet 8

Discussion topics: What is a sound wave? How do you derive the corresponding equation of motion? How is the speed of sound defined? What happens when the amplitude of the wave becomes large?

21. Stokes drift

An object floating in a pond / lake / sea / ocean on which surface waves are propagating experiences a net drift in the direction of wave propagation, called *Stokes drift*. This motion actually reflects the (average) motion of Lagrangian fluid particles, which is the topic of this exercise.

i. Let $\vec{\xi}(t,\vec{R})$ denote the displacement of a fluid particle which is at position $\vec{R}$ at the reference time t_0, i.e. $\vec{\xi}(t,\vec{R}) \equiv \vec{r}(t,\vec{R}) - \vec{r}(t_0,\vec{R})$, where $\vec{r}(t,\vec{R})$ denotes the Lagrangian trajectory. Show that the latter can be expressed in the form

$$\vec{r}(t,\vec{R}) = \vec{R} + \int_{t_0}^{t} \vec{\mathsf{v}}\big(t', \vec{R} + \vec{\xi}(t',\vec{R})\big)\,\mathrm{d}t', \tag{1}$$

where $\vec{\mathsf{v}}(t,\vec{r})$ denotes as usual the fluid velocity field. Deduce therefrom that for a small displacement one can write

$$\vec{r}(t,\vec{R}) = \vec{R} + \vec{\xi}(t,\vec{R}) + \int_{t_0}^{t} \big[\vec{\xi}(t',\vec{R})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t',\vec{R})\,\mathrm{d}t'. \tag{2}$$

ii. The displacement of fluid particles in an ocean (in the region $z \leq 0$) on which linear, gravity-induced waves are propagating in the x-direction, can be written as

$$\xi_x(t,x,z) = A\cos(kx-\omega t)\,\mathrm{e}^{kz} \quad , \quad \xi_z(t,x,z) = A\sin(kx-\omega t)\,\mathrm{e}^{kz}, \tag{3}$$

with A the wave amplitude.

a) Show that an object at $\vec{R} = (x,z)$ will drift into the x-direction with an average velocity

$$\vec{v}_{\text{drift}} = \omega k A^2\,\mathrm{e}^{2kz}\,\vec{\mathrm{e}}_x. \tag{4}$$

Note that this *Stokes drift velocity* is nonlinear in the wave amplitude.

b) Using the deep-ocean approximation for the dispersion relation of the surface waves, compute $|\vec{v}_{\text{drift}}|$ for waves with a period of 10 s and an amplitude of 1 m.

22. One-dimensional "similarity flow"

Consider a perfect fluid at rest in the region $x \geq 0$ with pressure $\mathcal{P}_0$ and mass density ρ_0; the region $x < 0$ is empty ($\mathcal{P} = 0$, $\rho = 0$). At time $t = 0$, the wall separating both regions is removed, so that the fluid starts flowing into the region $x < 0$. The goal of this exercise is to solve this instance of *Riemann's problem* by determining the flow velocity $\mathsf{v}(t,x)$ for $t > 0$. It will be assumed that the pressure and mass density of the fluid remain related by

$$\frac{\mathcal{P}}{\mathcal{P}_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \quad \text{with } \gamma > 1$$

throughout the motion. This relation also gives you the speed of sound $c_s(\rho)$.

i. Assume that the dependence on t and x of the various fields involves only the combination $u \equiv x/t$.[1] Show that the continuity and Euler equations can be recast as

$$\big[u - \mathsf{v}(u)\big]\,\rho'(u) = \rho(u)\,\mathsf{v}'(u)$$
$$\rho(u)\big[u - \mathsf{v}(u)\big]\,\mathsf{v}'(u) = c_s^2(\rho(u))\,\rho'(u),$$

where ρ' resp. v' denote the derivative of ρ resp. v with respect to u.

[1]... which is what is meant by "self-similar".

ii. Show that the velocity is either constant, or obeys the equation $u - \mathsf{v}(u) = c_s(\rho(u))$, in which case the squared speed of sound takes the form $c_s^2(\rho) = c_s^2(\rho_0)(\rho/\rho_0)^{\gamma-1}$.

iii. Show that the results of **i.** and **ii.** lead to the relation

$$\mathsf{v}(u) = a + \frac{2}{\gamma - 1}\, c_s(\rho(u)),$$

where a denotes a constant whose value is fixed by the condition that $\mathsf{v}(u)$ remain continuous inside the fluid. Show eventually that in some interval for the values of u, the norm of v is given by

$$|\mathsf{v}(u)| = \frac{2}{\gamma + 1}\big[c_s(\rho_0) - u\big].$$

iv. Sketch the profiles of the mass density $\rho(u)$ and the streamlines $x(t)$ and show that after the removal of the separation at $x = 0$ the information propagates with velocity $2c_s(\rho_0)/(\gamma - 1)$ towards the negative-x region, while it moves to the right with the speed of sound $c_s(\rho)$.

23. Inviscid Burgers equation

The purpose of this exercise is to show how an innocent-looking—yet non-linear—partial differential equation with a smooth initial condition may lead after finite amount of time to a discontinuity, i.e. a shock wave.

Neglecting the pressure term in the one-dimensional Euler equation leads to the so-called *inviscid Burgers equation*

$$\frac{\partial \mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x} = 0.$$

i. Show that the solution with (arbitrary) given initial condition $\mathsf{v}(0,x)$ for $x \in \mathbb{R}$ obeys the implicit equation $\mathsf{v}(0,x) = \mathsf{v}\big(t, x + \mathsf{v}(0,x)\,t\big)$.

Hint: `http://en.wikipedia.org/wiki/Burgers'_equation`

ii. Consider the initial condition $\mathsf{v}(0,x) = \mathsf{v}_0\,\mathrm{e}^{-(x/x_0)^2}$ with v_0 and x_0 two real numbers. Show that the flow velocity becomes discontinuous at time $t = \sqrt{\mathrm{e}/2}\,x_0/\mathsf{v}_0$, namely at $x = x_0\sqrt{2}$.