Tutorial sheet 7

19. Statics of rotating fluids

This exercise is strongly inspired by Chapter 13.3.3 of *Modern Classical Physics* by Roger D. Blandford and Kip S. Thorne.

Consider a fluid, bound by gravity, which is rotating rigidly, i.e. with a uniform angular velocity $\vec{\Omega}_0$ with respect to an inertial frame, around a given axis. In a reference frame that co-rotates with the fluid, the latter is at rest, and thus governed by the laws of hydrostatics—except that you now have to consider an additional term...

i. Relying on your knowledge from point mechanics, show that the usual equation of hydrostatics (in an inertial frame) is replaced in the co-rotating frame by

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\big[\Phi(\vec{r}) + \Phi_{\text{cen.}}(\vec{r})\big], \tag{1}$$

where $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big[\vec{\Omega}_0 \times \vec{r}\big]^2$ denotes the potential energy from which the centrifugal inertial force (density) derives, $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$, while $\Phi(\vec{r})$ is the gravitational potential energy.

ii. Show that Eq. (1) implies that the equipotential lines of $\Phi + \Phi_{\text{cen.}}$ coincide with the contours of constant mass density as well as with the isobars.

iii. Consider a slowly spinning fluid planet of mass M, assuming for the sake of simplicity that the mass is concentrated at the planet center, so that the gravitational potential is unaffected by the rotation. Let R_e resp. R_p denote the equatorial resp. polar radius of the planet, where $|R_e - R_p| \ll R_e \simeq R_p$, and g be the gravitational acceleration at the surface of the planet.
Using questions **i.** and **ii.**, show that the difference between the equatorial and polar radii is

$$R_e - R_p \simeq \frac{R_e^2|\vec{\Omega}_0|^2}{2g}.$$

Compute this difference in the case of Earth ($R_e \simeq 6.4 \times 10^3$ km)—which as everyone knows behaves as a fluid if you look at it long enough—and compare with the actual value.

20. Flow of a liquid in the vicinity of a gas bubble

We assume that the flow of the liquid is radial: $\vec{\mathsf{v}} = \mathsf{v}(t,r)\,\vec{e}_r$, where the gas bubble is assumed to sit at $\vec{r} = \vec{0}$. Throughout the exercise, the effect of the liquid-gas surface tension—which gives rise to a difference in pressure between both sides of the liquid-gas interface—is neglected.

i. a) Show that the liquid's flow is irrotational. (*Hint*: one can avoid the computation of the curl!)

b) Assuming in addition that the flow is incompressible, derive the expression of $\mathsf{v}(t,r)$ in terms of the bubble radius $R(t)$ and its derivative $\dot{R}(t)$. Deduce therefrom the velocity potential.

ii. One assumes that the gas inside the bubble is an ideal gas which evolves adiabatically when the bubble radius varies, i.e. that its pressure—assumed to be uniform—and volume obey $\mathcal{P}\mathcal{V}^\gamma = \text{constant}$, where γ is the heat capacity ratio. Let $\mathcal{P}_0$ be the value of the pressure at infinity and R_0 the bubble radius when the gas pressure equals $\mathcal{P}_0$.

a) Neglecting the gas flow, give the expression of the pressure inside the bubble in terms of the radius.

b) Writing the Euler equation in terms of the velocity potential, show that $R(t)$ obeys the evolution equation

$$\ddot{R}(t)R(t) + \frac{3\big[\dot{R}(t)\big]^2}{2} = \frac{\mathcal{P}_0}{\rho}\left[\left(\frac{R_0}{R(t)}\right)^{3\gamma} - 1\right], \tag{2}$$

where ρ is the liquid mass density.

iii. Suppose now that the bubble radius slightly oscillates about the equilibrium value R_0. Writing $R(t) = R_0[1 + \epsilon(t)]$ with $|\epsilon(t)| \ll 1$, derive the (linear!) evolution equation for $\epsilon(t)$. What is the frequency f of such small oscillations?

Numerical application: calculate f for air ($\gamma = 1.4$) bubbles with $R_0 = 1$ mm and $R_0 = 5$ mm in water ($\rho = 10^3$ kg/m^3) for $\mathcal{P}_0 = 10^5$ Pa.