Tutorial sheet 5

Discussion topic: What is a potential flow? What are the corresponding equations of motion?

14. Hill's spherical vortex

An *incompressible* steady flow in the whole space $\mathbb{R}^3$ is such that the associated vorticity field reads

$$\vec{\omega}(r,\theta,\varphi) = \frac{\omega}{a}\vec{\mathrm{e}}_\varphi \quad \text{for } r \le a \tag{1}$$

and $\vec{\omega} = 0$ for $r > a$, with $\omega > 0$, $a > 0$. That is, the vorticity is distributed inside a sphere of radius a and vanishing outside. Note that $\vec{\mathrm{e}}_\varphi$ denotes the "natural" basis vector associated with φ, i.e. such that

$$\mathrm{d}\vec{r} = \vec{\mathrm{e}}_r\,\mathrm{d}r + \vec{\mathrm{e}}_\theta\,\mathrm{d}\theta + \vec{\mathrm{e}}_\varphi\,\mathrm{d}\varphi.$$

The goal of this exercise is to determine the corresponding flow velocity $\vec{\mathsf{v}}(\vec{r}) = \mathsf{v}^r(r,\theta)\,\vec{\mathrm{e}}_r + \mathsf{v}^\theta(r,\theta)\,\vec{\mathrm{e}}_\theta$.

i. The geometry of the problem begs for the use of spherical coordinates (r,θ,φ), everything being independent of the azimuthal angle φ. Give (determine?) the expressions of the divergence and the curl of the velocity field $\vec{\mathsf{v}}(r,\theta)$, and of the gradient and the Laplacian of a scalar function $\Phi(r,\theta)$.

ii. Inside the sphere, $r \le a$

a) Show that the components of the velocity obey the system of equations

$$\begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\mathsf{v}^r) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(\mathsf{v}^\theta\sin\theta) = 0 \\ \dfrac{1}{r^2\sin\theta}\left(\dfrac{\partial\mathsf{v}^\theta}{\partial r} - \dfrac{\partial\mathsf{v}^r}{\partial\theta}\right) = \dfrac{\omega}{a}. \end{cases} \tag{2}$$

b) To look for a solution of the system (2), one makes the ansatz $\mathsf{v}^r = f(r)\cos\theta$, $\mathsf{v}^\theta = g(r)\sin\theta$. Show that the system allows you to express $g(r)$ in terms of $f(r)$, and that the latter obeys the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[r^2 f(r)\right] - 2f(r) = -\frac{2\omega r^2}{a}. \tag{3}$$

c) For which values of $\alpha \in \mathbb{Z}$ do you have solutions of Eq. (3) of the form $f(r) = C_\alpha r^\alpha$ with $C_\alpha \in \mathbb{R}$? For one of these values of α, the constant C_α is determined by Eq. (3).

In turn, the other two integration constants C_α have to be determined by physical arguments: $\mathsf{v}^r = 0$ at $r = a$ (why?) and another condition at $r = 0$ (which one?). Deduce from these conditions the form of v^r, then of v^θ inside the sphere.

Remark: you should find that v^θ is nonvanishing at $r = a$.

ii. Outside the sphere

... the flow is irrotational, so that you can introduce a velocity potential $\Phi(r,\theta)$.

a) Show that the latter is of the form

$$\Phi(r,\theta) = \left(\frac{A'}{r} + \frac{B'}{r^2}\right)\cos\theta$$

and determine the constants A', B' by requesting the continuity of the velocity field at the sphere boundary.

b) Determine the velocity field $\vec{\mathsf{v}}(r,\theta)$. How does it behaves as $r \to \infty$? What does this means for the vortex, as seen from an observer comoving with the fluid at infinity?

15. Pointlike source

Consider the fluid motion defined in a system of cylindrical coordinates (r, θ, z) by the velocity field given for $r \neq 0$ by

$$\mathsf{v}^r(t,\vec{r}) = \frac{f(t)}{r}, \quad \mathsf{v}^\theta(t,\vec{r}) = 0, \quad \mathsf{v}^z(t,\vec{r}) = 0, \tag{4}$$

with f some scalar function.

i. Calculate the strain rate tensor; what are its principal axes? Give the volume expansion rate. Compute the vorticity vector.

Remark You may of course work with Cartesian coordinates; but you should give a try at working directly in cylindrical coordinates, to apply the ideas of the lecture(s) on curvilinear coordinates.

ii. Mathematically, the velocity field is singular at $r = 0$. Thinking of the velocity profile, what do you have *physically* at that point if $f(t) > 0$? if $f(t) < 0$?