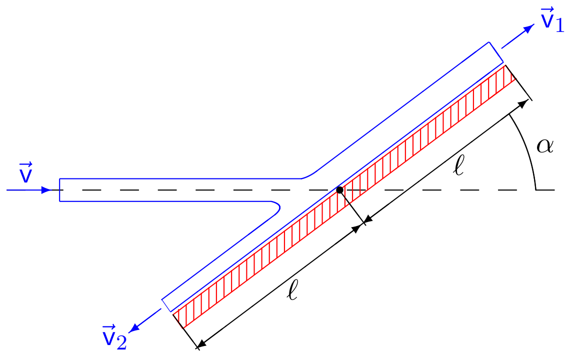
Tutorial sheet 4

Discussion topics:

- What is the Bernoulli equation? Give some examples of application.
- What is Kelvin's circulation theorem? What does it imply for the vorticity?

10. Water jet

A horizontal jet of water with cross section area $\mathcal{S} = 20\ \mathrm{cm}^2$ and velocity $\mathsf{v} = 20\,\mathrm{m}\cdot\mathrm{s}^{-1}$ hits an inclined board of length $2\ell = 20$ cm making an angle α with the horizontal direction, and splits into two jets 1 and 2. The resulting flow is assumed to be steady and incompressible, and water is modeled as a perfect fluid.



i. Show that the influence of gravity on the velocities v_1, v_2 is negligible, so that you can forget it when applying the equation appropriate for the flow under study (which you should apply at the water/air boundary).

ii. Knowing that the force $\vec{F}$ exerted by the water on the board is normal to the latter (why?), determine the cross-section areas $\mathcal{S}_1$, $\mathcal{S}_2$ of the jets as functions of $\mathcal{S}$ and the angle α.

iii. Determine the force $\vec{F}$ and compute the numerical value of $|\vec{F}|$ for $\alpha = 30^{\rm o}$.

11. Differential form of Thomson's theorem

i. Consider the motion of a perfect barotropic fluid with conservative external volume forces. Show that the vorticity vector field $\vec{\omega}(t,\vec{r})$ obeys the equation

$$\frac{\partial\vec{\omega}(t,\vec{r})}{\partial t} = \vec{\nabla}\times\big[\vec{\mathsf{v}}(t,\vec{r})\times\vec{\omega}(t,\vec{r})\big].$$

ii. Stationary vortex: Let $\vec{\omega}(t,\vec{r}) = A\,\delta(x^1)\,\delta(x^2)\,\vec{\mathrm{e}}_3$ be the vorticity field in a fluid, with A a real constant and $\{x^i\}$ Cartesian coordinates. Determine the corresponding flow velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

Hint: You should invoke symmetry arguments and Stokes' theorem. A useful formal analogy is provided by the Maxwell–Ampère equation of magnetostatics.

12. Model of a tornado

In a simplified approach, one may model a tornado as the steady incompressible flow of a perfect fluid—air—with mass density $\rho = 1.3\ \mathrm{kg}\cdot\mathrm{m}^{-3}$, with a vorticity $\vec{\omega}(\vec{r}) = \omega(\vec{r})\,\vec{\mathrm{e}}_3$ which remains uniform inside a cylinder—the "eye" of the tornado—with (vertical) axis along $\vec{\mathrm{e}}_3$ and a finite radius $a = 50$ m, and vanishes outside.

i. Express the velocity $\mathsf{v}(r) \equiv \left|\vec{\mathsf{v}}(\vec{r})\right|$ at a distance $r = |\vec{r}|$ from the axis as a function of r and and the velocity $\mathsf{v}_a \equiv \mathsf{v}(r{=}a)$ at the edge of the eye.
Compute ω inside the eye, assuming $\mathsf{v}_a = 180$ km/h.

ii. Show that for $r > a$ the tornado is equivalent to a vortex at $x^1 = x^2 = 0$ (as in exercise **11.ii**). What is the circulation around a closed curve circling this equivalent vortex?

iii. Assuming that the pressure $\mathcal{P}$ far from the tornado equals the "normal" atmospheric pressure $\mathcal{P}_0$, determine $\mathcal{P}(r)$ for $r > a$. Compute the barometric depression $\Delta\mathcal{P} \equiv \mathcal{P}_0 - \mathcal{P}$ at the edge of the eye.
Consider a horizontal roof made of a material with mass surface density 100 kg/m^2: is it endangered by the tornado?

13. Vortex sheet

Consider a flow for which the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{\mathsf{v}}$ is large in a thin layer of thickness δ. If the product $\delta \cdot \vec{\omega}$ remains finite—and converges towards a vector $\vec{\varpi}$ in the plane *tangent* to the layer—when $\delta \to 0$, the surface to which the layer shrinks in that limit is referred to as *vortex sheet*.

Prove that if some surface $\mathcal{S}$ is either a vortex sheet, or a surface at which the tangential component of the velocity is discontinuous, then $\vec{\varpi} \times \vec{e}_n = [\![\vec{\mathsf{v}}_{||}]\!]$ with $\vec{e}_n$ the (local) unit normal vector to $\mathcal{S}$ and $[\![\vec{\mathsf{v}}_{||}]\!]$ the (local) jump of the velocity component tangential to $\mathcal{S}$. Consequently, a vortex sheet is a surface of tangential discontinuity of the velocity, and reciprocally.
Vortex sheets arise for example in the flow around the wing of an airplane