Tutorial sheet 2

Discussion topics:

- What are the Lagrangian and Eulerian descriptions? How is a fluid defined?
- What are the strain rate tensor, the rotation rate tensor, and the vorticity vector? How do they come about and what do they measure?

3. Stationary flow: second example

Consider the fluid flow whose velocity field $\vec{\mathsf{v}}(t,\vec{r})$ has coordinates (in a given Cartesian system)

$$\mathsf{v}^1(t,\vec{r}) = kx^2, \quad \mathsf{v}^2(t,\vec{r}) = kx^1, \quad \mathsf{v}^3(t,\vec{r}) = 0, \tag{1}$$

where k is a positive real number, while x^1, x^2, x^3 are the coordinates of the position vector $\vec{r}$.

i. Determine the stream lines at an arbitrary instant t.

ii. Let X^1, X^2, X^3 denote the coordinates of some arbitrary point M and let t_0 be the real number defined by

$$kt_0 = \begin{cases} -\mathrm{Artanh}(X^2/X^1) & \text{if } |X^1| > |X^2| \\ 0 & \text{if } \ X^1 = \pm X^2 \\ -\mathrm{Artanh}(X^1/X^2) & \text{if } |X^1| < |X^2|. \end{cases}$$

Write down a parameterization $x^1(t)$, $x^2(t)$, $x^3(t)$, in terms of a parameter denoted by t, of the coordinates of *the* stream line $\vec{x}(t)$ going through M such that $\mathrm{d}\vec{x}(t)/\mathrm{d}t$ at any point *equals* the velocity field at that point, and that either $x^1(t) = 0$ or $x^2(t) = 0$ for $t = t_0$.

iii. Viewing $\vec{x}(t)$ as the trajectory of a point—actually, of a fluid particle—, you already know the velocity of that point at time t (do you?). What is its acceleration $\vec{a}(t)$?

iv. Coming back to the velocity field (1), compute first its partial derivative $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$, then the *material derivative*

$$\frac{\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})}{\mathrm{D}t} \equiv \frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\vec{\mathsf{v}}(t,\vec{r}).$$

Compare $\partial\vec{\mathsf{v}}(t,\vec{r})/\partial t$ and $\mathrm{D}\vec{\mathsf{v}}(t,\vec{r})/\mathrm{D}t$ with the acceleration of a fluid particle found in question **iii.**

4. Lagrangian description: Jacobian determinant

Consider the twice continuously differentiable ($\mathscr{C}^2$) mapping $(t,\vec{R}) \mapsto \vec{r}(t,\vec{R})$ from "initial" position vectors at t_0 to those at time t. Let (X^1,X^2,X^3) resp. (x^1,x^2,x^3) denote the coordinates of $\vec{R}$ resp. $\vec{r}$ in some fixed system.

The *Jacobian determinant* $J(t,\vec{R})$ of the transformation $\vec{R} \mapsto \vec{r}$ is as usual the determinant of the matrix with elements $\partial x^i/\partial X^j$. Thanks to the hypotheses on the mapping $\vec{r}(t,\vec{R})$, this Jacobian has simple mathematical properties.

i. Can you find a physical interpretation for $J(t,\vec{R})$? [*Hint*: Think of small volume elements.]

ii. Using the initial value $J(t_0,\vec{R})$ in the reference configuration, as well as the invertibility and $\mathscr{C}^2$-character of the mapping $\vec{r}(t,\vec{R})$, show that $J(t,\vec{R})$ is positive for $t \geq t_0$. What does this mean physically?

iii. Consider the motion of a continuous medium defined for $t \geq 0$ by

$$x^1 = X^1 + ktX^2, \quad x^2 = X^2 + ktX^1, \quad x^3 = X^3,$$

where $k > 0$. One may for simplicity assume that the coordinates are Cartesian.

a) Over which time range is this motion defined? [*Hint*: Jacobian determinant!]

b) What are its pathlines?

c) Determine the Eulerian description of this motion, i.e. the velocity field $\vec{\mathsf{v}}(t, \vec{r})$.

5. Isotropy of pressure

Consider a geometrical point at position $\vec{r}$ in a fluid at rest. The stress vector across every surface element going through this point is normal: $\vec{T}(\vec{r}) = -\mathcal{P}(\vec{r})\,\vec{\mathrm{e}}_n$, with $\vec{\mathrm{e}}_n$ the unit vector orthogonal to the surface element under consideration. Show that the (hydrostatic) pressure $\mathcal{P}$ is independent of the orientation of $\vec{\mathrm{e}}_n$.

Hint: Consider the forces on the faces of an infinitesimal trirectangular tetrahedron.