Tutorial sheet 13

36. A family of solutions of the dynamical equations for perfect relativistic fluids

Let $\{x^\mu\}$ denote Minkowski coordinates and $\tau^2 \equiv -x^\mu x_\mu$, where the "mostly plus" metric is used. Show that the following four-velocity, pressure and charge density constitute a solution of the equations describing the motion of a perfect relativistic fluid with equation of state $\mathcal{P} = K\varepsilon$ and a single conserved charge:

$$u^\mu(\mathsf{x}) = \frac{x^\mu}{\tau} \quad , \quad \mathcal{P}(\mathsf{x}) = \mathcal{P}_0\left(\frac{\tau_0}{\tau}\right)^{3(1+K)} \quad , \quad n(\mathsf{x}) = n_0\left(\frac{\tau_0}{\tau}\right)^3 \mathcal{N}\big(\sigma(\mathsf{x})\big), \tag{1}$$

with τ_0, $\mathcal{P}_0$, n_0 arbitrary constants and $\mathcal{N}$ an arbitrary function of a single argument, while σ is a function of spacetime coordinates with vanishing comoving derivative: $u^\mu\partial_\mu\sigma(\mathsf{x}) = 0$.

37. Speed of sound in ultrarelativistic matter

Consider a perfect fluid with the usual energy-momentum tensor. $T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (\epsilon + \mathcal{P})u^\mu u^\nu/c^2$. It is assumed that there is no conserved quantum number relevant for thermodynamics, so that the energy density in the local rest frame ϵ is function of a single thermodynamic variable, for instance $\epsilon = \epsilon(\mathcal{P})$. Throughout the exercise, Minkowski coordinates are used.

A background "flow" with uniform local-rest-frame energy density and pressure ϵ_0 and $\mathcal{P}_0$ is submitted to a small perturbation resulting in $\epsilon = \epsilon_0 + \delta\epsilon$, $\mathcal{P} = \mathcal{P}_0 + \delta\mathcal{P}$, and $\vec{\mathsf{v}} = \vec{0} + \delta\vec{\mathsf{v}}$.

i. Starting from the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, show that linearization to first order in the perturbations leads to the two equations of motion $\partial_t\delta\epsilon = -(\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\delta\vec{\mathsf{v}}$ and $(\epsilon_0 + \mathcal{P}_0)\partial_t\delta\vec{\mathsf{v}} = -c^2\vec{\nabla}\delta\mathcal{P}$.

ii. Show that the speed of sound is given by the expression $c_s^2 = \dfrac{c^2}{\mathrm{d}\epsilon/\mathrm{d}\mathcal{P}}$.

iii. Compute c_s for a fluid obeying the Stefan–Boltzmann law[1] $\mathcal{P} = \dfrac{g\pi^2}{90}\dfrac{(k_\mathrm{B}T)^4}{(\hbar c)^3}$, with g the number of degrees of freedom (e.g. $g = 2$ for blackbody radiation).

Hint: You may find the Gibbs–Duhem relation useful...

38. One-dimensional relativistic flow

In the July 3rd lecture, the equations describing the "boost-invariant" one-dimensional expansion of a perfect relativistic fluid were presented. Here, we investigate another one-dimensional solution of the equations of relativistic fluid dynamics, originally due to L. Landau himself [Izv. Akad. Nauk. USSR **17** (1953) 51], again for a medium without conserved quantum number.

Throughout the exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

i. Considering a one-dimensional expansion along the z-axis, write down the non-trivial equations of motion expressing energy-momentum conservation in Minkowski coordinates.
From now on, the equation of state of the expanding perfect fluid is assumed to be $\epsilon = 3\mathcal{P}$.

[1] This is a good opportunity to refresh your knowledge on the statistical physics of relativistic systems. Can you give a physical argument why quantum effects always play a role in such systems, as signaled by the presence of $\hbar$ in the equation of state?

ii. The so-called *light-cone coordinates* are defined as $x^+ \equiv \dfrac{t+z}{\sqrt{2}}$, $x^- \equiv \dfrac{t-z}{\sqrt{2}}$, where the factor $1/\sqrt{2}$ is not universal, yet convenient.

a) Check the identities $\dfrac{\partial}{\partial t} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x^+} + \dfrac{\partial}{\partial x^-}\right)$ and $\dfrac{\partial}{\partial z} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x^+} - \dfrac{\partial}{\partial x^-}\right)$.

b) Show that they allow you to transform the equations of motion of question **i.** into

$$\frac{\partial \epsilon}{\partial x^+} + 2\frac{\partial(\epsilon \mathrm{e}^{-2y_\mathrm{f}})}{\partial x^-} = 0 \quad , \quad 2\frac{\partial(\epsilon \mathrm{e}^{2y_\mathrm{f}})}{\partial x^+} + \frac{\partial \epsilon}{\partial x^-} = 0, \tag{2}$$

with $y_\mathrm{f}(\mathsf{x})$ the position-dependent “flow rapidity”, such that $u^t = \cosh y_\mathrm{f}$, $u^z = \sinh y_\mathrm{f}$.

iii. Let $y_\pm \equiv \log \dfrac{x^\pm}{\Delta}$, with Δ some length scale.

a) Show that the expansion with energy density and flow rapidity

$$\epsilon(y_+, y_-) = \epsilon_0 \exp\left[-\frac{4}{3}\left(y_+ + y_- - \sqrt{y_+ y_-}\right)\right] \quad , \quad y_\mathrm{f}(y_+, y_-) = \frac{y_+ - y_-}{2} \tag{3}$$

is solution to the equations of motion (2), with ϵ_0 a constant.

Hint: The identity $\mathrm{e}^{2y_\mathrm{f}(y_+,y_-)} = x^+/x^-$ may be helpful.

b) Transforming the previous solution (3) back to Minkowski variables, sketch the energy density (in units of ϵ_0) and the flow rapidity as a function of z at successive instants $t = \Delta$, 2Δ, 4Δ, 8Δ for $|z| < t$. Can you guess what physical problem Landau was trying to model?

iv. If you still have time... This is irrelevant for the rest of the exercise, yet can you write down the metric tensor in light-cone coordinates? Of which type are the basis vectors in the x^+ and x^- directions?