Tutorial sheet 11

Discussion topic: Dynamical similarity and the Reynolds number. You could also educate yourself on the topic of Life at low Reynolds number and the "scallop theorem" by reading E. M. Purcell's article.

30. Dimensionless equations of motion for sea surface waves

This exercise is partly a continuation of the lecture (May 23) on linear sea surface waves, which you should check if you are not sure of the notations employed.

The equations of motion governing gravity waves at the free surface of an incompressible perfect liquid (ocean/sea water) in a gravity field $-g\,\vec{e}_z$ are

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0, \tag{1a}$$

$$\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\,\vec{\mathsf{v}}(t,\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(t,\vec{r}) - g\,\vec{e}_z, \tag{1b}$$

with the boundary conditions $\mathsf{v}_z(t,x,z\!=\!0) = 0$ at the sea bottom;

$$\mathsf{v}_z\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \frac{\partial\delta h(t,x)}{\partial t} + \mathsf{v}_x(t,\vec{r})\frac{\partial\delta h(t,x)}{\partial x} \tag{1c}$$

at the free surface, situated at $z = h_0 + \delta h(t,x)$; and a uniform pressure at that same free surface, which may be re-expressed as

$$\mathcal{P}\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \rho g\,\delta h(t,x) + \mathcal{P}_0 \tag{1d}$$

with $\mathcal{P}_0$ a constant whose precise value is irrelevant. As in the lecture, the problem is assumed to be two-dimensional.

i. We introduce characteristic scales for various quantities: δh_c for the amplitude of the surface deformation; L_c for lengths along the horizontal direction x; and t_c for durations—in practice, the "good" choice would be $t_c = L_c/c_s$ with c_s the speed of sound, yet this is irrelevant here. With their help, we define dimensionless variables

$$t^* \equiv \frac{t}{t_c}, \quad x^* \equiv \frac{x}{L_c}, \quad z^* \equiv \frac{z}{L_c},$$

and fields:

$$\delta h^* \equiv \frac{\delta h}{\delta h_c}, \quad \mathsf{v}_x^* \equiv \frac{\mathsf{v}_x}{\delta h_c/t_c}, \quad \mathsf{v}_z^* \equiv \frac{\mathsf{v}_z}{\delta h_c/t_c}, \quad \mathcal{P}^* \equiv \frac{\mathcal{P}-\mathcal{P}_0}{\rho\,\delta h_c L_c/t_c^2}.$$

Considering the latter as functions of the reduced variables t^*, x^*, z^*, rewrite the equations (1a)–(1d), making use of the dimensionless numbers

$$\mathrm{Fr} \equiv \frac{\sqrt{L_c/g}}{t_c}, \quad \varepsilon \equiv \frac{\delta h_c}{L_c}, \quad \delta \equiv \frac{h_0}{L_c}.$$

What does the parameter ε control (mathematically)? and the parameter δ (physically)?

ii. Assuming that the flow is irrotational, show that you can combine some of the dimensionless equations found in question **i.** into

$$\frac{\partial\mathsf{v}_x^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^*\frac{\partial\mathsf{v}_x^*}{\partial x^*} + \mathsf{v}_z^*\frac{\partial\mathsf{v}_z^*}{\partial x^*}\right) + \frac{1}{\mathrm{Fr}^2}\frac{\partial\delta h^*}{\partial x^*} = 0. \tag{2}$$

The various equations you have obtained in this exercise will be exploited later in the lecture to derive the *Korteweg–de Vries equation*, which governs in a specific limit the evolution of the function $\phi(t^*,x^*) \equiv \delta h^*(t^*,x^*)/\delta$, i.e. the profile of the free water surface.

31. (1+1)-dimensional relativistic motion

On June 27th, the flow velocities considered in the lectures will reach the relativistic regime. To prepare for this event, you may refresh your knowledge on Special Relativity. This exercise is here to help you in that direction, and also introduces coordinates which will be used later in the lectures.

Consider a (1+1)-dimensional relativistic motion along a direction denoted as x, where "1+1" stands for one time and one spatial dimension. A first possibility is to use Minkowski $(x^0, x^1) = (t, x)$ coordinates, such that the metric tensor has components $g_{00} \equiv g_{tt} = -1$, $g_{11} \equiv g_{xx} = +1$, $g_{01} = g_{10} = 0$.[1] If there is a high-velocity motion in the x-direction, a better choice might be to used the *proper time* (of a comoving observer) τ and *spatial rapidity* ς such that

$$x^{0'} \equiv \tau \equiv \sqrt{t^2 - x^2}, \quad x^{1'} \equiv \varsigma \equiv \frac{1}{2} \log \frac{t+x}{t-x} \quad \text{where } |x| \leq t. \tag{3}$$

Throughout, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein's summation convention.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau \cosh \varsigma, \quad x = \tau \sinh \varsigma. \tag{4}$$

(*Hint:* Recognize $\frac{1}{2} \log \frac{1+u}{1-u}$).
Deduce the following relationship between the basis vectors of the two coordinate systems

$$\begin{cases} \vec{e}_\tau(\tau, \varsigma) = \cosh \varsigma \, \vec{e}_t + \sinh \varsigma \, \vec{e}_x \\ \vec{e}_\varsigma(\tau, \varsigma) = \tau \sinh \varsigma \, \vec{e}_t + \tau \cosh \varsigma \, \vec{e}_x \end{cases} \tag{5}$$

and write down the metric tensor $g_{0'0'} \equiv g_{\tau\tau}$, $g_{1'1'} \equiv g_{\varsigma\varsigma} \dots$ in the new coordinate system.
For the sake of completeness, give also the components $g^{\mu'\nu'}$ of the inverse metric tensor.

ii. Inspiring yourself from what was done in the case of the two-dimensional Euclidean plane in the lecture, compute the Christoffel symbols $\Gamma^{\mu'}_{\nu'\rho'}$ where the primed indices run over all values $0'$, $1'$.

iii. Let N^μ denote the components of a "2-vector".
Write down the covariant derivative $\mathrm{d}_{\nu'} N^{\mu'} \equiv \mathrm{d}N^{\mu'}/\mathrm{d}x^{\nu'}$ that generalizes to curvilinear (τ, ς) coordinates the derivative $\partial_\nu N^\mu \equiv \partial N^\mu / \partial x^\nu$ of Minkowski coordinates. Compute the "2-divergence" $\mathrm{d}_{\mu'} N^{\mu'}$.

iv. Let $T^{\mu\nu}$ denote the components of a symmetric $\binom{2}{0}$-tensor, such that $T^{01} = 0$.
Write down the covariant derivative $\mathrm{d}_{\rho'} T^{\mu'\nu'}$ and compute $\mathrm{d}_{\mu'} T^{\mu'\nu'}$ for $\nu' \in \{\tau, \varsigma\}$.

v. Draw on a spacetime diagram—with t on the vertical axis and x on the horizontal axis—the lines of constant τ and those of constant ς.

Remark: The coordinates (τ, σ) are sometimes called *Milne coordinates*.

[1] Note that I shall use the "mostly plus" convention for the metric tensor, in which timelike vectors have a negative semi-norm.