Tutorial sheet 9

26. Equations of motion of a perfect relativistic fluid

In this exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

i. Show that the tensor with components $\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$ defines a projector on the subspace orthogonal to the 4-velocity.
Denoting by d_μ the components of the (covariant) 4-gradient, we define $\nabla^\nu \equiv \Delta^{\mu\nu}\mathrm{d}_\mu$. Can you see the rationale behind this notation?

ii. Show that the energy-momentum conservation equation for a perfect fluid is equivalent to the two equations

$$u^\mu \mathrm{d}_\mu \epsilon + (\epsilon + \mathcal{P})\mathrm{d}_\mu u^\mu = 0 \quad \text{and} \quad (\epsilon + \mathcal{P})u^\mu \mathrm{d}_\mu u^\nu + \nabla^\nu \mathcal{P} = 0.$$

Which known equation does the second one evoke?

27. Particle number conservation

Consider a 4-current with components $N^\mu(x)$ obeying the continuity equation $\mathrm{d}_\mu N^\mu(\mathsf{x}) = 0$. Show that the quantity $\mathcal{N} = \int N^0(\mathsf{x})\,\mathrm{d}^3\vec{r}/c$ is a Lorentz scalar, by convincing yourself first that $\mathcal{N}$ can be rewritten in the form

$$\mathcal{N} = \frac{1}{c}\int_{x^0=\text{const.}} N^\mu(\mathsf{x})\,\mathrm{d}^3\sigma_\mu, \tag{1}$$

where $\mathrm{d}^3\sigma_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ is a 4-vector, with $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ the antisymmetric 4-tensor defined by

$$\mathrm{d}^3\mathcal{V}^{012} = \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2, \quad \mathrm{d}^3\mathcal{V}^{021} = -\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^1, \quad \text{etc.}$$

and $\epsilon_{\mu\nu\rho\sigma}$ the totally antisymmetric Levi–Civita tensor with the convention $\epsilon_{0123} = +1$, such that $\mathrm{d}^3\mathcal{V}^{\nu\rho\sigma}$ represents the 3-dimensional hypersurface element in Minkowski space.

28. Speed of sound in ultrarelativistic matter

Consider a perfect fluid with the usual energy-momentum tensor. $T^{\mu\nu} = \mathcal{P}g^{\mu\nu} + (\epsilon + \mathcal{P})u^\mu u^\nu/c^2$. It is assumed that there is no conserved particle number relevant for thermodynamics, so that the energy density in the local rest frame ϵ is function of a single thermodynamic variable, for instance $\epsilon = \epsilon(\mathcal{P})$. Throughout the exercise, Minkowski coordinates are used.

A background "flow" with local-rest-frame energy density and pressure ϵ_0 and $\mathcal{P}_0$ is submitted to a small perturbation resulting in $\epsilon = \epsilon_0 + \delta\epsilon$, $\mathcal{P} = \mathcal{P}_0 + \delta\mathcal{P}$, and $\vec{\mathsf{v}} = \vec{0} + \delta\vec{\mathsf{v}}$.

i. Starting from the energy-momentum conservation equation $\partial_\mu T^{\mu\nu} = 0$, show that linearization to first order in the perturbations leads to the two equations of motion $\partial_t\delta\epsilon = -(\epsilon_0 + \mathcal{P}_0)\vec{\nabla}\cdot\delta\vec{\mathsf{v}}$ and $(\epsilon_0 + \mathcal{P}_0)\partial_t\delta\vec{\mathsf{v}} = -c^2\vec{\nabla}\delta\mathcal{P}$.

ii. Show that the speed of sound is given by the expression $c_s^2 = \dfrac{c^2}{\mathrm{d}\epsilon/\mathrm{d}\mathcal{P}}$.

iii. Compute c_s for a fluid obeying the Stefan–Boltzmann law[1] $\mathcal{P} = \dfrac{g\pi^2}{90}\dfrac{(k_\mathrm{B}T)^4}{(\hbar c)^3}$, with g the number of degrees of freedom (e.g. $g = 2$ for blackbody radiation).

Hint: You may find the Gibbs–Duhem relation useful...

[1] This is a good opportunity to refresh your knowledge on the statistical physics of relativistic systems. Can you give a physical argument why quantum effects always play a role in such systems, as signaled by the presence of $\hbar$ in the equation of state?