Tutorial sheet 7

Discussion topics: What is a sound wave? How do you derive the corresponding equation of motion? How is the speed of sound defined? What happens when the amplitude of the wave becomes large?

20. One-dimensional "similarity flow"

Consider a perfect fluid at rest in the region $x \geq 0$ with pressure $\mathcal{P}_0$ and mass density ρ_0; the region $x < 0$ is empty ($\mathcal{P} = 0$, $\rho = 0$). At time $t = 0$, the wall separating both regions is removed, so that the fluid starts flowing into the region $x < 0$. The goal of this exercise is to solve this instance of *Riemann's problem* by determining the flow velocity $\mathsf{v}(t, x)$ for $t > 0$. It will be assumed that the pressure and mass density of the fluid remain related by

$$\frac{\mathcal{P}}{\mathcal{P}_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \qquad \text{with } \gamma > 1$$

throughout the motion. This relation also gives you the speed of sound $c_s(\rho)$.

i. Assume that the dependence on t and x of the various fields involves only the combination $u \equiv x/t$.[1] Show that the continuity and Euler equations can be recast as

$$\big[u - \mathsf{v}(u)\big]\, \rho'(u) = \rho(u)\, \mathsf{v}'(u)$$
$$\rho(u)\big[u - \mathsf{v}(u)\big]\, \mathsf{v}'(u) = c_s^2(\rho(u))\, \rho'(u),$$

where ρ' resp. v' denote the derivative of ρ resp. v with respect to u.

ii. Show that the velocity is either constant, or obeys the equation $u - \mathsf{v}(u) = c_s(\rho(u))$, in which case the squared speed of sound takes the form $c_s^2(\rho) = c_s^2(\rho_0)(\rho/\rho_0)^{\gamma-1}$.

iii. Show that the results of **i.** and **ii.** lead to the relation

$$\mathsf{v}(u) = a + \frac{2}{\gamma - 1}\, c_s(\rho(u)),$$

where a denotes a constant whose value is fixed by the condition that $\mathsf{v}(u)$ remain continuous inside the fluid. Show eventually that in some interval for the values of u, the norm of v is given by

$$|\mathsf{v}(u)| = \frac{2}{\gamma + 1}\big[c_s(\rho_0) - u\big].$$

iv. Sketch the profiles of the mass density $\rho(u)$ and the streamlines $x(t)$ and show that after the removal of the separation at $x = 0$ the information propagates with velocity $2c_s(\rho_0)/(\gamma - 1)$ towards the negative-x region, while it moves to the right with the speed of sound $c_s(\rho)$.

21. Inviscid Burgers equation

The purpose of this exercise is to show how an innocent-looking—yet non-linear—partial differential equation with a smooth initial condition may lead after finite amount of time to a discontinuity, i.e. a shock wave.

Neglecting the pressure term in the one-dimensional Euler equation leads to the so-called *inviscid Burgers equation*

$$\frac{\partial \mathsf{v}(t,x)}{\partial t} + \mathsf{v}(t,x)\frac{\partial \mathsf{v}(t,x)}{\partial x} = 0.$$

i. Show that the solution with (arbitrary) given initial condition $\mathsf{v}(0, x)$ for $x \in \mathbb{R}$ obeys the implicit equation $\mathsf{v}(0, x) = \mathsf{v}\big(t, x + \mathsf{v}(0, x)\, t\big)$.

Hint: `http://en.wikipedia.org/wiki/Burgers'_equation`

[1]... which is what is meant by "self-similar".

ii. Consider the initial condition $\mathsf{v}(0,x) = \mathsf{v}_0\,\mathrm{e}^{-(x/x_0)^2}$ with v_0 and x_0 two real numbers. Show that the flow velocity becomes discontinuous at time $t = \sqrt{\mathrm{e}/2}\,x_0/\mathsf{v}_0$, namely at $x = x_0\sqrt{2}$.

22. (1+1)-dimensional relativistic motion

On May 28th, the flow velocities considered in the lectures will reach the relativistic regime. To prepare for this event, you may refresh your knowledge on Special Relativity. This exercise is here to help you in that direction, and also introduces coordinates which will be used later in the lectures.

Consider a (1+1)-dimensional relativistic motion along a direction denoted as x, where "1+1" stands for one time and one spatial dimension. A first possibility is to use Minkowski $(x^0, x^1) = (t, x)$ coordinates, such that the metric tensor has components $g_{00} \equiv g_{tt} = -1$, $g_{11} \equiv g_{xx} = +1$, $g_{01} = g_{10} = 0$.[2] If there is a high-velocity motion in the x-direction, a better choice might be to used the *proper time* (of a comoving observer) τ and *spatial rapidity* ς such that

$$x^{0'} \equiv \tau \equiv \sqrt{t^2 - x^2}, \quad x^{1'} \equiv \varsigma \equiv \frac{1}{2}\log\frac{t+x}{t-x} \quad \text{where } |x| \leq t.$$

Throughout, we use a system of units in which the speed of light in vacuum c equals 1, as well as Einstein's summation convention.

i. Check that the relations defining τ and ς can be inverted, yielding the much simpler

$$t = \tau\cosh\varsigma, \quad x = \tau\sinh\varsigma.$$

(*Hint:* Recognize $\frac{1}{2}\log\frac{1+u}{1-u}$).

Deduce the following relationship between the basis vectors of the two coordinate systems

$$\begin{cases} \vec{\mathrm{e}}_\tau(\tau,\varsigma) = \cosh\varsigma\,\vec{\mathrm{e}}_t + \sinh\varsigma\,\vec{\mathrm{e}}_x \\ \vec{\mathrm{e}}_\varsigma(\tau,\varsigma) = \tau\sinh\varsigma\,\vec{\mathrm{e}}_t + \tau\cosh\varsigma\,\vec{\mathrm{e}}_x \end{cases}$$

and write down the metric tensor $g_{0'0'} \equiv g_{\tau\tau}$, $g_{1'1'} \equiv g_{\varsigma\varsigma}\dots$ in the new coordinate system.
For the sake of completeness, give also the components $g^{\mu'\nu'}$ of the inverse metric tensor.

ii. Inspiring yourself from what was done in the case of the two-dimensional Euclidean plane in the lecture, compute the Christoffel symbols $\Gamma^{\mu'}_{\nu'\rho'}$ where the primed indices run over all values $0'$, $1'$.

iii. Let N^μ denote the components of a "2-vector".
Write down the covariant derivative $\mathrm{d}_{\nu'}N^{\mu'} \equiv \mathrm{d}N^{\mu'}/\mathrm{d}x^{\nu'}$ that generalizes to curvilinear (τ,ς) coordinates the derivative $\partial_\nu N^\mu \equiv \partial N^\mu/\partial x^\nu$ of Minkowski coordinates. Compute the "2-divergence" $\mathrm{d}_{\mu'}N^{\mu'}$.

iv. Let $T^{\mu\nu}$ denote the components of a symmetric $\binom{2}{0}$-tensor, such that $T^{01} = 0$.
Write down the covariant derivative $\mathrm{d}_{\rho'}T^{\mu'\nu'}$ and compute $\mathrm{d}_{\mu'}T^{\mu'\nu'}$ for $\nu' \in \{\tau,\varsigma\}$.

v. Draw on a spacetime diagram—with t on the vertical axis and x on the horizontal axis—the lines of constant τ and those of constant ς.

Remark: The coordinates (τ,σ) are sometimes called *Milne coordinates*.

[2] Note that I shall use the "mostly plus" convention for the metric tensor, in which timelike vectors have a negative semi-norm.