Tutorial sheet 6

Discussion topic: What is a potential flow? What are the corresponding equations of motion?

17. Statics of rotating fluids

This exercise is strongly inspired by Chapter 13.3.3 of the lecture notes on *Applications of Classical Physics* by Roger D. Blandford and Kip S. Thorne.

Consider a fluid, bound by gravity, which is rotating rigidly, i.e. with a uniform angular velocity $\vec{\Omega}_0$ with respect to an inertial frame, around a given axis. In a reference frame that co-rotates with the fluid, the latter is at rest, and thus governed by the laws of hydrostatics—except that you now have to consider an additional term...

i. Relying on your knowledge from point mechanics, show that the usual equation of hydrostatics (in an inertial frame) is replaced in the co-rotating frame by

$$\frac{1}{\rho(\vec{r})}\vec{\nabla}\mathcal{P}(\vec{r}) = -\vec{\nabla}\big[\Phi(\vec{r}) + \Phi_{\text{cen.}}(\vec{r})\big], \tag{1}$$

where $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big[\vec{\Omega}_0 \times \vec{r}\big]^2$ denotes the potential energy from which the centrifugal inertial force (density) derives, $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$, while $\Phi(\vec{r})$ is the gravitational potential energy.

ii. Show that Eq. (1) implies that the equipotential lines of $\Phi + \Phi_{\text{cen.}}$ coincide with the contours of constant mass density as well as with the isobars.

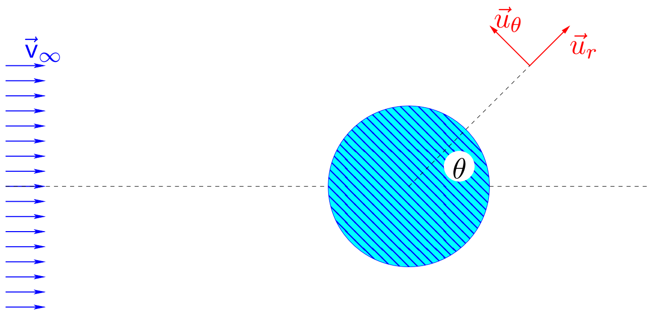
iii. Consider a slowly spinning fluid planet of mass M, assuming for the sake of simplicity that the mass is concentrated at the planet center, so that the gravitational potential is unaffected by the rotation. Let R_e resp. R_p denote the equatorial resp. polar radius of the planet, where $|R_e - R_p| \ll R_e \simeq R_p$, and g be the gravitational acceleration at the surface of the planet.
Using questions **i.** and **ii.**, show that the difference between the equatorial and polar radii is

$$R_e - R_p \simeq \frac{R_e^2|\vec{\Omega}_0|^2}{2g}.$$

Compute this difference in the case of Earth ($R_e \simeq 6.4 \times 10^3$ km)—which as everyone knows behaves as a fluid if you look at it long enough—and compare with the actual value.

18. Potential flow with a vortex. Magnus effect

The purpose of this exercise is to introduce a simplified model for the Magnus effect, which was discussed in the lectures.



One can show (cf. Landau–Lifshitz, *Fluid dynamics*, § 10) that the flow velocity of an incompressible perfect fluid around a cylinder of radius R at rest, with the uniform condition $\vec{\mathsf{v}}(\vec{r}) = \vec{\mathsf{v}}_\infty$ far from the cylinder—$\vec{\mathsf{v}}_\infty$ being perpendicular to the cylinder axis—, is given by

$$\vec{\mathsf{v}}(r,\theta) = \mathsf{v}_\infty \left[\left(1 - \frac{R^2}{r^2}\right)\cos\theta\,\vec{u}_r - \left(1 + \frac{R^2}{r^2}\right)\sin\theta\,\vec{u}_\theta\right], \tag{2}$$

where (r,θ) are polar coordinates—the third dimension (z), along the cylinder axis, plays no role—with the origin at the center of the cylinder (see Figure) and $\vec{u}_r$, $\vec{u}_\theta$ unit length vectors.

One superposes to the velocity field (2) a vortex with circulation Γ, corresponding to a flow velocity

$$\vec{\mathsf{v}}(r,\theta) = \frac{\Gamma}{2\pi r}\vec{u}_\theta. \tag{3}$$

i. Let $C \equiv \Gamma/(4\pi R \mathsf{v}_\infty)$. Determine the points with vanishing velocity for the flow resulting from superposing (2) and (3).

Hint: Distinguish the two cases $C < 1$ and $C > 1$.

ii. How do the streamlines look like in each case? Comment on the physical meaning of the result.

iii. Express the force per unit length $\mathrm{d}\vec{F}/\mathrm{d}z$ exerted on the cylinder by the flow (2)+(3) as function of Γ, v_∞ and the mass density ρ of the fluid.

19. Two-dimensional potential flow. Teapot effect

dedicated mit freundlichen Grüßen to T.L., for whom this exercise was the highlight of the lectures.

Consider a two-dimensional potential flow with velocity $\vec{\mathsf{v}}(t,x,y)$, with (x,y) Cartesian coordinates. Let $\varphi(t,x,y)$ be the corresponding velocity potential ($\vec{\mathsf{v}} = -\vec{\nabla}\varphi$) and $\psi(t,x,y)$ the so-called "stream function", such that $\mathsf{v}_x = -\partial_y\psi$ and $\mathsf{v}_y = \partial_x\psi$. Define a complex variable z by $z = x + \mathrm{i}y$.

i. Show that the complex potential defined by $\phi = \varphi + \mathrm{i}\psi$ is a holomorphic/analytic function of z, by checking that the Cauchy–Riemann equations hold.

ii. Show that the stream function obeys the Laplace differential equation and that the lines of constant $\psi(t,x,y)$ are the streamlines of the flow.

iii. Check that the "complex velocity" $\mathsf{w} \equiv -\dfrac{\mathrm{d}\phi}{\mathrm{d}z}$ equals $\mathsf{v}_x - \mathrm{i}\mathsf{v}_y$.

iv. Consider the complex potential $\phi(z) = Az^n$ with $A \in \mathbb{R}$ and $n \geq 1/2$. Show that this potential allows you to describe the flow in the sector $\widehat{\mathcal{E}}$ delimited by two walls making an angle $\alpha = \pi/n$.

Hint: Landau–Lifshitz, *Fluid dynamics*, § 10.

v. What can you say about the flow velocity in the vicinity of the end-corner of the sector $\widehat{\mathcal{E}}$?

Hint: Distinguish the cases $\alpha < \pi$ and $\alpha > \pi$.

vi. Teapot effect

If one tries to pour tea "carefully" from a teapot, one will observe that the liquid will trickle along the lower side of the nozzle, instead of falling down into the cup waiting below. Explain this phenomenon using the flow profile introduced above (in the case $\alpha > \pi$) and the Bernoulli equation.

Literature: Jearl Walker, Scientific American, Oct. 1984 (= Spektrum der Wissenschaft, Feb. 1985).

vii. Assuming now that you are using the potential $\phi(z) = Az^n$ to model the flow of a river, which qualitative behavior can you anticipate for its bank?