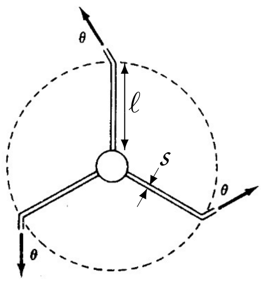
Tutorial sheet 5

Discussion topic: What is Kelvin's circulation theorem? What does it imply for the vorticity?

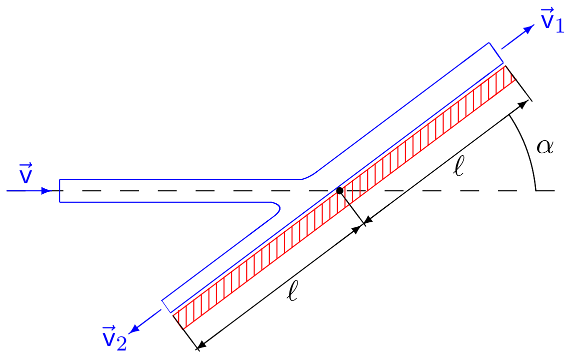
13. Water sprinkler

The horizontal lawn sprinkler schematized below is fed water through its center with a mass flow rate Q. Assuming that water is a perfect incompressible fluid, determine the steady rotation rate as function of Q, the cross section area s of the pipes, their length ℓ, and the angle θ of the emerging water jets with respect to the respective pipes.



14. Water jet

A horizontal jet of water with cross section area $\mathcal{S} = 20$ cm^2 and velocity $\mathsf{v} = 20\,\mathrm{m\cdot s^{-1}}$ hits an inclined board of length $2\ell = 20$ cm making an angle α with the horizontal direction, and splits into two jets 1 and 2. The resulting flow is assumed to be steady and incompressible, and water is modeled as a perfect fluid.



i. Show that the influence of gravity on the velocities v_1, v_2 is negligible, so that you can forget it when applying the equation appropriate for the flow under study (which you should apply at the water/air boundary).

ii. Knowing that the force $\vec{F}$ exerted by the water on the board is normal to the latter (why?), determine the cross-section areas $\mathcal{S}_1$, $\mathcal{S}_2$ of the jets as functions of $\mathcal{S}$ and the angle α.

iii. Determine the force $\vec{F}$ and compute the numerical value of $|\vec{F}|$ for $\alpha = 30^{\rm o}$.

15. Model of a tornado

In a simplified approach, one may model a tornado as the steady incompressible flow of a perfect fluid—air—with mass density $\rho = 1.3\ \mathrm{kg\cdot m^{-3}}$, with a vorticity $\vec{\omega}(\vec{r}) = \omega(\vec{r})\,\vec{\mathrm{e}}_3$ which remains uniform inside a cylinder—the "eye" of the tornado—with (vertical) axis along $\vec{\mathrm{e}}_3$ and a finite radius $a = 50$ m, and vanishes outside.

i. Express the velocity $\mathsf{v}(r) \equiv |\vec{\mathsf{v}}(\vec{r})|$ at a distance $r = |\vec{r}|$ from the axis as a function of r and and the velocity $\mathsf{v}_a \equiv \mathsf{v}(r = a)$ at the edge of the eye.
Compute ω inside the eye, assuming $\mathsf{v}_a = 180$ km/h.

ii. Show that for $r > a$ the tornado is equivalent to a vortex at $x^1 = x^2 = 0$ (as in exercise **11.ii**). What is the circulation around a closed curve circling this equivalent vortex?

iii. Assuming that the pressure $\mathcal{P}$ far from the tornado equals the "normal" atmospheric pressure $\mathcal{P}_0$, determine $\mathcal{P}(r)$ for $r > a$. Compute the barometric depression $\Delta\mathcal{P} \equiv \mathcal{P}_0 - \mathcal{P}$ at the edge of the eye.
Consider a horizontal roof made of a material with mass surface density 100 kg/m^2: is it endangered by the tornado?

16. Vortex sheet

Consider a flow for which the vorticity $\vec{\omega} = \vec{\nabla} \times \vec{\mathsf{v}}$ is large in a thin layer of thickness δ. If the product $\delta \cdot \vec{\omega}$ remains finite—and converges towards a vector $\vec{\varpi}$ in the plane *tangent* to the layer—when $\delta \to 0$, the surface to which the layer shrinks in that limit is referred to as *vortex sheet*.

Prove that if some surface $\mathcal{S}$ is either a vortex sheet, or a surface at which the tangential component of the velocity is discontinuous, then $\vec{\varpi} \times \vec{e}_n = [\![\vec{\mathsf{v}}_{||}]\!]$ with $\vec{e}_n$ the (local) unit normal vector to $\mathcal{S}$ and $[\![\vec{\mathsf{v}}_{||}]\!]$ the (local) jump of the velocity component tangential to $\mathcal{S}$. Consequently, a vortex sheet is a surface of tangential discontinuity of the velocity, and reciprocally.
Vortex sheets arise for example in the flow around the wing of an airplane