Tutorial sheet 2

Discussion topics: What are the Lagrangian and Eulerian descriptions? How is a fluid defined?

4. Lagrangian description: Jacobian determinant

Consider the twice continuously differentiable ($\mathscr{C}^2$) mapping $(t,\vec{R}) \mapsto \vec{r}(t,\vec{R})$ from "initial" position vectors at t_0 to those at time t. Let (X^1, X^2, X^3) resp. (x^1, x^2, x^3) denote the coordinates of $\vec{R}$ resp. $\vec{r}$ in some fixed system.

The *Jacobian determinant* $J(t,\vec{R})$ of the transformation $\vec{R} \mapsto \vec{r}$ is as usual the determinant of the matrix with elements $\partial x^i/\partial X^j$. Thanks to the hypotheses on the mapping $\vec{r}(t,\vec{R})$, this Jacobian has simple mathematical properties.

i. Can you find a physical interpretation for $J(t,\vec{R})$? [*Hint*: Think of small volume elements.]

ii. Using the initial value $J(t_0,\vec{R})$ in the reference configuration, as well as the invertibility and $\mathscr{C}^2$-character of the mapping $\vec{r}(t,\vec{R})$, show that $J(t,\vec{R})$ is positive for $t \geq t_0$. What does this mean physically?

iii. Consider the motion of a continuous medium defined for $t \geq 0$ by

$$x^1 = X^1 + ktX^2, \quad x^2 = X^2 + ktX^1, \quad x^3 = X^3,$$

where $k > 0$. One may for simplicity assume that the coordinates are Cartesian.

a) Over which time range is this motion defined? [*Hint*: Jacobian determinant!]

b) What are its pathlines?

c) Determine the Eulerian description of this motion, i.e. the velocity field $\vec{\mathsf{v}}(t,\vec{r})$.

5. Stress tensor

Let $\mathbf{T}_{ij}$ denote the Cartesian[1] components of the stress tensor in a continuous medium. Consider an infinitesimal cube of medium of side $\mathrm{d}\ell$, whose sides are parallel to the axes of the coordinate system.

i. Explain why the k-component $\mathcal{M}_k$ of the torque exerted on the cube by the neighboring regions of the continuous medium obeys $\mathcal{M}_k \propto -\epsilon_{ijk}\mathbf{T}_{ij}(\mathrm{d}\ell)^3$, with ϵ_{ijk} the usual Levi-Civita symbol.

ii. Using dimensional considerations, write down the dependence of the moment of inertia I of the cube on $\mathrm{d}\ell$ and on the continuum mass density ρ.

iii. Using the results of the previous two questions, how does the rate of change of the angular velocity ω_k scale with $\mathrm{d}\ell$? How can you prevent this rate of change from diverging in the limit $\mathrm{d}\ell \to 0$?

6. Isotropy of pressure

Consider a geometrical point at position $\vec{r}$ in a fluid at rest. The stress vector across every surface element going through this point is normal: $\vec{T}(\vec{r}) = -\mathcal{P}(\vec{r})\,\vec{\mathrm{e}}_n$, with $\vec{\mathrm{e}}_n$ the unit vector orthogonal to the surface element under consideration. Show that the (hydrostatic) pressure $\mathcal{P}$ is independent of the orientation of $\vec{\mathrm{e}}_n$.

Hint: Consider the forces on the faces of an infinitesimal trirectangular tetrahedron.

[1]... which allows us to be sloppy with the position of indices.