Tutorial sheet 12

Discussion topic: Life at low Reynolds number (educate yourself by reading E. M. Purcell's article) and the "scallop theorem".

35. Equations of fluid dynamics in a uniformly rotating reference frame

This exercise is inspired by Chapter 14.5.1 of the lecture notes on *Applications of Classical Physics* by Roger D. Blandford and Kip S. Thorne.

For the study of various physical problems (see examples in question **iv.a**), it may be more convenient to study the dynamics of a fluid from a reference frame $\mathcal{R}_{\Omega_0}$ in uniform rotation with angular velocity $\vec{\Omega}_0$ with respect to an inertial frame $\mathcal{R}_0$.
In exercise **17**, you already investigated hydrostatics in a rotating reference frame: in that case only the centrifugal acceleration plays a role, which can be entirely recast as the effect of a potential energy $\Phi_{\text{cen.}}(\vec{r}) \equiv -\frac{1}{2}\big(\vec{\Omega}_0 \times \vec{r}\big)^2$ leading to the centrifugal inertial force density $\vec{f}_{\text{cen.}} = -\rho\vec{\nabla}\Phi_{\text{cen.}}$. The purpose of this exercise is to generalize that result to the derivation of (some of) the equations governing a flowing Newtonian fluid.

i. Kinematics

Recall the expressions of the centrifugal and Coriolis accelerations acting on a small fluid element in terms of its position vector $\vec{r}$ and velocity $\vec{\mathsf{v}}$ (measured in $\mathcal{R}_{\Omega_0}$) and of the angular velocity.

ii. Incompressibility condition

Writing down the relation between the velocity $\vec{\mathsf{v}}$ with respect to $\mathcal{R}_{\Omega_0}$ and that measured in $\mathcal{R}_0$, show that the incompressibility condition valid in the inertial frame leads to $\vec{\nabla}\cdot\vec{\mathsf{v}} = 0$.

iii. Navier–Stokes equation

Show that the incompressible Navier–Stokes equation from the point of view of an observer at rest in the rotating reference frame $\mathcal{R}_{\Omega_0}$ reads (the variables are omitted)

$$\frac{\mathrm{D}\vec{\mathsf{v}}}{\mathrm{D}t} = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}_{\text{eff.}} + \nu\triangle\vec{\mathsf{v}} - 2\vec{\Omega}_0\times\vec{\mathsf{v}} \tag{1}$$

where $\mathcal{P}_{\text{eff.}} = \mathcal{P} + \rho\big(\Phi + \Phi_{\text{cen.}}\big)$, with Φ the potential energy from which (non-inertial) volume forces acting on the fluid derive. Check that you recover the equation of hydrostatics found in exercise **17**.

iv. Dimensionless numbers and limiting cases

a) Let L_c resp. v_c denote a characteristic length resp. velocity for a given flow. The Ekman and Rossby numbers are respectively defined as

$$\text{Ek} \equiv \frac{\nu}{|\Omega_0| L_c^2} \qquad , \qquad \text{Ro} \equiv \frac{\mathsf{v}_c}{|\Omega_0| L_c}.$$

Compute Ek and Ro in a few numerical examples:
– $L_c \approx 100$ km, $\mathsf{v}_c \approx 10\ \mathrm{m\cdot s^{-1}}$, $\Omega_0 \approx 10^{-4}\ \mathrm{rad\cdot s^{-1}}$, $\nu \approx 10^{-5}\ \mathrm{m^2\cdot s^{-1}}$ (wind in the Earth atmosphere);
– $L_c \approx 1000$ km, $\mathsf{v}_c \approx 0.1\ \mathrm{m\cdot s^{-1}}$, $\Omega_0 \approx 10^{-4}\ \mathrm{rad\cdot s^{-1}}$, $\nu \approx 10^{-6}\ \mathrm{m^2\cdot s^{-1}}$ (ocean stream);
– $L_c \approx 10$ cm, $\mathsf{v}_c \approx 1\ \mathrm{m\cdot s^{-1}}$, $\Omega_0 \approx 10\ \mathrm{rad\cdot s^{-1}}$, $\nu \approx 10^{-6}\ \mathrm{m^2\cdot s^{-1}}$ (coffee/tea in your cup).

b) Assuming stationarity, which term in Eq. (1) is negligible (against which) at small Ekman number? at small Rossby number?

Write down the simplified equation of motion valid when both Ek $\ll 1$ and Ro $\ll 1$ (to which of the above examples does this correspond?). How do the (effective) pressure gradient $\vec{\nabla}\mathcal{P}_{\text{eff.}}$ and flow velocity stand relative to each other?

36. Dimensionless equations of motion for sea surface waves

This exercise is partly a continuation of the May 26 lecture on linear sea surface waves, which you should check if you are not sure of the notations employed.

The equations of motion governing gravity waves at the free surface of an incompressible perfect liquid (ocean/sea water) in a gravity field $-gz\,\vec{e}_z$ are

$$\vec{\nabla}\cdot\vec{\mathsf{v}}(t,\vec{r}) = 0, \tag{2a}$$

$$\frac{\partial\vec{\mathsf{v}}(t,\vec{r})}{\partial t} + \big[\vec{\mathsf{v}}(t,\vec{r})\cdot\vec{\nabla}\big]\,\vec{\mathsf{v}}(t,\vec{r}) = -\frac{1}{\rho}\vec{\nabla}\mathcal{P}(t,\vec{r}) - g\,\vec{e}_z, \tag{2b}$$

with the boundary conditions $\mathsf{v}_z(t,x,z\!=\!0) = 0$ at the sea bottom;

$$\mathsf{v}_z\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \frac{\partial\delta h(t,x)}{\partial t} + \mathsf{v}_x(t,\vec{r})\frac{\partial\delta h(t,x)}{\partial x} \tag{2c}$$

at the free surface, situated at $z = h_0 + \delta h(t,x)$; and a uniform pressure at that same free surface, which may be re-expressed as

$$\mathcal{P}\big(t,x,z\!=\!h_0\!+\!\delta h(t,x)\big) = \rho g\,\delta h(t,x) + \mathcal{P}_0 \tag{2d}$$

with $\mathcal{P}_0$ a constant whose precise value is irrelevant. As in the lecture (May 26), the problem is assumed to be two-dimensional.

i. We introduce characteristic scales for various quantities: δh_c for the amplitude of the surface deformation; L_c for lengths along the horizontal direction x; and t_c for durations—in practice, the "good" choice would be $t_c = L_c/c_s$ with c_s the speed of sound, yet this is irrelevant here. With their help, we define dimensionless variables

$$t^* \equiv \frac{t}{t_c},\quad x^* \equiv \frac{x}{L_c},\quad z^* \equiv \frac{z}{L_c},$$

and fields:

$$\delta h^* \equiv \frac{\delta h}{\delta h_c},\quad \mathsf{v}_x^* \equiv \frac{\mathsf{v}_x}{\delta h_c/t_c},\quad \mathsf{v}_z^* \equiv \frac{\mathsf{v}_z}{\delta h_c/t_c},\quad \mathcal{P}^* \equiv \frac{\mathcal{P}-\mathcal{P}_0}{\rho\,\delta h_c L_c/t_c^2}.$$

Considering the latter as functions of the reduced variables t^*, x^*, z^*, rewrite the equations (2a)–(2d), making use of the dimensionless numbers

$$\mathrm{Fr} \equiv \frac{\sqrt{L_c/g}}{t_c},\quad \varepsilon \equiv \frac{\delta h_c}{L_c},\quad \delta \equiv \frac{h_0}{L_c}.$$

What does the parameter ε control (mathematically)? and the parameter δ (physically)?

ii. Assuming that the flow is irrotational, show that you can combine some of the dimensionless equations found in question **i.** into

$$\frac{\partial\mathsf{v}_x^*}{\partial t^*} + \varepsilon\left(\mathsf{v}_x^*\frac{\partial\mathsf{v}_x^*}{\partial x^*} + \mathsf{v}_z^*\frac{\partial\mathsf{v}_z^*}{\partial x^*}\right) + \frac{1}{\mathrm{Fr}^2}\frac{\partial\delta h^*}{\partial x^*} = 0.$$

The various equations you have obtained in this exercise will be exploited later in the lecture, to derive the *Korteweg–de Vries equation*, which governs in a specific limit the evolution of the function $\phi(t^*,x^*) \equiv \delta h^*(t^*,x^*)/\delta$, i.e. the profile of the free water surface.