Tutorial sheet 11

Discussion topic: Dynamical similarity and the Reynolds number

32. Flow due to an oscillating plane boundary

Consider a rigid infinitely extended plane boundary ($y = 0$) that oscillates in its own plane with a sinusoidal velocity $U\cos(\omega t)\,\vec{e}_x$. The region $y > 0$ is filled with an incompressible Newtonian fluid with uniform kinematic shear viscosity ν. We shall assume that volume forces on the fluid are negligible, that the pressure is uniform and remains constant in time, and that the fluid motion induced by the plane oscillations do not depend on the coordinates x, z.

i. Determine the flow velocity $\vec{\mathsf{v}}(t, y)$ and plot the resulting profile.

ii. What is the characteristic thickness of the fluid layer in the vicinity of the plane boundary that follows the oscillations? Comment on your result.

33. Dimensional consideration for viscous flows in a tube

Consider the motion of a given fluid in a cylindrical tube of length L and of circular cross section under the action of a difference $\Delta\mathcal{P}$ between the pressures at the two ends of the tube. The relation between the pressure drop per unit length $\Delta\mathcal{P}/L$ and the magnitude of the mean velocity $\langle\mathsf{v}\rangle$—defined as the average over a cross section of the tube—is given by

$$\frac{\Delta\mathcal{P}}{L} = C\langle\mathsf{v}\rangle^n,$$

with C a constant that depends on the fluid mass density ρ, on the kinematic shear viscosity ν, and on the radius a of the tube cross section. n is a number which depends on the type of flow: $n = 1$ if the flow is laminar (this is the Hagen–Poiseuille law seen in the lecture), while measurements in turbulent flows by Hagen (1854) resp. Reynolds (1883) have given $n = 1.75$ resp. $n = 1.722$.

Assuming that C is—up to a pure number—a product of powers of ρ, ν and a, determine the exponents of these power laws using dimensional arguments.

34. Laminar flow of a river

A river is modeled as an incompressible Newtonian fluid flowing steadily and laminarly down a channel—the river bed—with uniform rectangular cross-section inclined at a constant angle α from the horizontal. The river itself is assumed to be a layer of constant thickness h, so that its free surface is a plane parallel to its bottom.

To fix notations, let x denote the direction along which water flows, with the basis vector oriented downstream, and y be the direction perpendicular to the river bed, oriented upwards. The river bottom resp. free surface is at $y = 0$ resp. $y = h$, the vertical edges of the river bed are at $z = \pm b$ with $2b$ the river width.

i. Equations of motion

Assuming that the pressure at the free surface of the water as well as "at the ends" at large $|x|$ is constant—i.e., the river flow is caused by gravity, not by a pressure gradient—, show that the flow velocity magnitude v, pressure $\mathcal{P}$ and mass density ρ of the fluid obey the equations

$$\begin{cases} \dfrac{\partial\mathsf{v}}{\partial x} = 0 \\ \eta\triangle\mathsf{v} = -\rho g\sin\alpha \\ \dfrac{\partial\mathcal{P}}{\partial y} = -\rho g\cos\alpha, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} \mathsf{v} = 0 & \text{at } y = 0 \text{ and at } z = \pm b \\ \dfrac{\partial \mathsf{v}}{\partial y} = 0 & \text{at } y = h \\ \mathcal{P} = \mathcal{P}_0 & \text{at } y = h. \end{cases} \tag{2}$$

You can immediately write down the solution for the pressure.

ii. Flow velocity

For the sake of brevity, the constant $-(\rho g/\eta)\sin\alpha$ will from now on be denoted by c.

a) Check that the ansatz

$$\mathsf{v}(\vec{r}) = \sum_{n=0}^{\infty} f_n(y)\cos(k_n z) \text{ with } k_n \equiv (2n+1)\frac{\pi}{2b} \tag{3}$$

automatically fulfills some of the equations and boundary conditions.

b) The physics we are interested is restricted to the region $-b \leq z \leq b$, so that we can extend any function arbitrarily beyond the river bed edges. Accordingly, a neat trick is to continue the constant c to a periodic—as suggested by ansatz (3)—non-constant function, namely the sum

$$\tilde{c}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{2\,c}{k_n b}\cos(k_n z). \tag{4}$$

Check (plot!) that this function does coincide with c on the interval $-b < z < b$. Do you know how it looks like outside the interval?

c) Write down the (inhomogeneous) differential equation obeyed by each of the functions $f_n(y)$. Solve it under consideration of the as yet unused boundary conditions (2) at $y = 0$ and $y = h$.

Normally, you should find $\mathsf{v}(\vec{r}) = \sum_{n=0}^{\infty} (-1)^n \frac{2\,c}{k_n^3 b}\left[\frac{\cosh\big(k_n(y-h)\big)}{\cosh(k_n h)} - 1\right]\cos(k_n z)$.

d) Show the mass flow rate across the river cross section is

$$Q = 4c\left(\sum_{n=0}^{\infty} \frac{\tanh(k_n h)}{k_n^5 b} - \frac{h}{b}\sum_{n=0}^{\infty} \frac{1}{k_n^4}\right).$$

Assuming that the river is much wider than deep, compute the leading order ($\propto bh^3$) contribution to the mass flow rate.

Remark: This leading order coincides with the result in the case where the river bed is unbounded in the z-direction, see Landau & Lifshitz, *Fluid Mechanics*, § 17 Problem 5. The next-to-leading order contribution is unfortunately much harder to obtain, since the Taylor expansion of the tanh term starts giving divergent series. In a 1910 article by a student of Arnold Sommerfeld—who also thanks Peter Debye in his acknowledgments—one may find an expression for this next-to-leading order term, which however seems very dubious, as it is proportional to h^4, while the expression of Q is odd in h. . .