Tutorial sheet 10

Discussion topic: What are the fundamental equations governing the dynamics of non-relativistic Newtonian fluids?

29. One-dimensional relativistic flow

In the lecture on the 9th of June, the equations describing the "boost-invariant" one-dimensional expansion of a perfect relativistic fluid were presented. Here, we investigate another one-dimensional solution of the equations of relativistic fluid dynamics, originally due to L. Landau [Izv. Akad. Nauk. USSR **17** (1953) 51], again for a medium without conserved quantum number.

Throughout the exercise, we set $c = 1$ and drop the x variable for the sake of brevity. Remember that the metric tensor has signature $(-,+,+,+)$.

i. Considering a one-dimensional expansion along the z-axis, write down the non-trivial equations of motion expressing energy-momentum conservation in Minkowski coordinates.
From now on, the equation of state of the expanding perfect fluid is assumed to be $\epsilon = 3\mathcal{P}$.

ii. The so-called *light-cone coordinates* are defined as $x^+ \equiv \dfrac{t+z}{\sqrt{2}}$, $x^- \equiv \dfrac{t-z}{\sqrt{2}}$, where the factor $1/\sqrt{2}$ is not universal, yet convenient.

a) Although this is irrelevant for the rest of the exercise, write down the metric tensor in light-cone coordinates. Of which type are the basis vectors in the x^+ and x^- directions?

b) Check the identities $\dfrac{\partial}{\partial t} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x^+} + \dfrac{\partial}{\partial x^-}\right)$ and $\dfrac{\partial}{\partial z} = \dfrac{1}{\sqrt{2}}\left(\dfrac{\partial}{\partial x^+} - \dfrac{\partial}{\partial x^-}\right)$.

c) Show that they allow you to transform the equations of motion of question **i.** into

$$\frac{\partial \epsilon}{\partial x^+} + 2\frac{\partial(\epsilon \mathrm{e}^{-2y_\mathrm{f}})}{\partial x^-} = 0 \quad , \quad 2\frac{\partial(\epsilon \mathrm{e}^{2y_\mathrm{f}})}{\partial x^+} + \frac{\partial \epsilon}{\partial x^-} = 0, \tag{1}$$

with $y_\mathrm{f}(\mathsf{x})$ the position-dependent "flow rapidity", such that $u^t = \cosh y_\mathrm{f}$, $u^z = \sinh y_\mathrm{f}$.

iii. Let $y_\pm \equiv \log \dfrac{x^\pm}{\Delta}$, with Δ some length scale.

a) Show that the expansion with energy density and flow rapidity

$$\epsilon(y_+, y_-) = \epsilon_0 \exp\left[-\frac{4}{3}\left(y_+ + y_- - \sqrt{y_+ y_-}\right)\right] \quad , \quad y_\mathrm{f}(y_+, y_-) = \frac{y_+ - y_-}{2} \tag{2}$$

is solution to the equations of motion (1), with ϵ_0 a constant.
Hint: The identity $\mathrm{e}^{2y_\mathrm{f}(y_+,y_-)} = x^+/x^-$ may be helpful.

b) Transforming the previous solution (2) back to Minkowski variables, sketch the energy density (in units of ϵ_0) and the flow rapidity as a function of z at successive instants $t = \Delta, 2\Delta, 4\Delta, 8\Delta$ for $|z| < t$. Can you guess what physical problem Landau was trying to model?

30. Heat diffusion

In the lecture (June 11), we derived the equation

$$\frac{\partial e(t,\vec{r})}{\partial t} = \vec{\nabla} \cdot \left[\kappa(t,\vec{r})\vec{\nabla}T(t,\vec{r})\right]$$

valid in a dissipative fluid at rest, with κ the heat capacity.

Assuming that $C \equiv \partial e/\partial T$ and κ are constant coefficients and introducing $\chi \equiv \kappa/C$, determine the temperature profile $T(t,\vec{r})$ for $z < 0$ with the boundary condition of a uniform temperature in the plane $z = 0$, which evolves in time as $T(t, z\!=\!0) = T_0 \cos(\omega t)$. At which depth is the amplitude of the temperature oscillations $10\,\%$ of that in the plane $z = 0$?

31. Taylor–Couette flow. Measurement of shear viscosity

A Couette viscometer consists of an annular gap, filled with fluid, between two concentric cylinders with height L. The outer cylinder (radius R_2) rotates around the common axis with angular velocity Ω_2, while the inner cylinder (radius R_1) remains motionless. The motion of the fluid is assumed to be two-dimensional, incompressible and steady.

i. Check that the continuity equation leads to $\mathsf{v}^r = 0$, with v^r the radial component (in a system of cylinder coordinates) of the flow velocity.

ii. Prove that the Navier–Stokes equation lead to the equations

$$\frac{\mathsf{v}^\varphi(r)^2}{r} = \frac{1}{\rho}\frac{\partial \mathcal{P}(r)}{\partial r} \tag{3}$$

$$\frac{\partial^2 \mathsf{v}^\varphi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \mathsf{v}^\varphi(r)}{\partial r} - \frac{\mathsf{v}^\varphi(r)}{r^2} = 0. \tag{4}$$

What is the meaning of Eq. (3)? Solve Eq. (4) with the ansatz $\mathsf{v}^\varphi(r) = ar + \dfrac{b}{r}$.

iii. One can show (can you?) that the $r\varphi$-component of the stress tensor is given by

$$\sigma^{r\varphi} = \eta\left(\frac{1}{r}\frac{\partial \mathsf{v}^r}{\partial \varphi} + \frac{\partial \mathsf{v}^\varphi}{\partial r} - \frac{\mathsf{v}^\varphi}{r}\right).$$

Show that $\sigma^{r\varphi} = -\dfrac{2b\eta}{r^2}$, where b is the same coefficient as above.

iv. A torque $\mathcal{M}_z$ is measured at the surface of the inner cylinder. How can the shear viscosity η of the fluid be deduced from this measurement?
Numerical example: $R_1 = 10$ cm, $R_2 = 11$ cm, $L = 10$ cm, $\Omega_2 = 10$ rad·s^{-1} and $\mathcal{M}_z = 7,246 \cdot 10^{-3}$ N·m.